Error and fallacy in control

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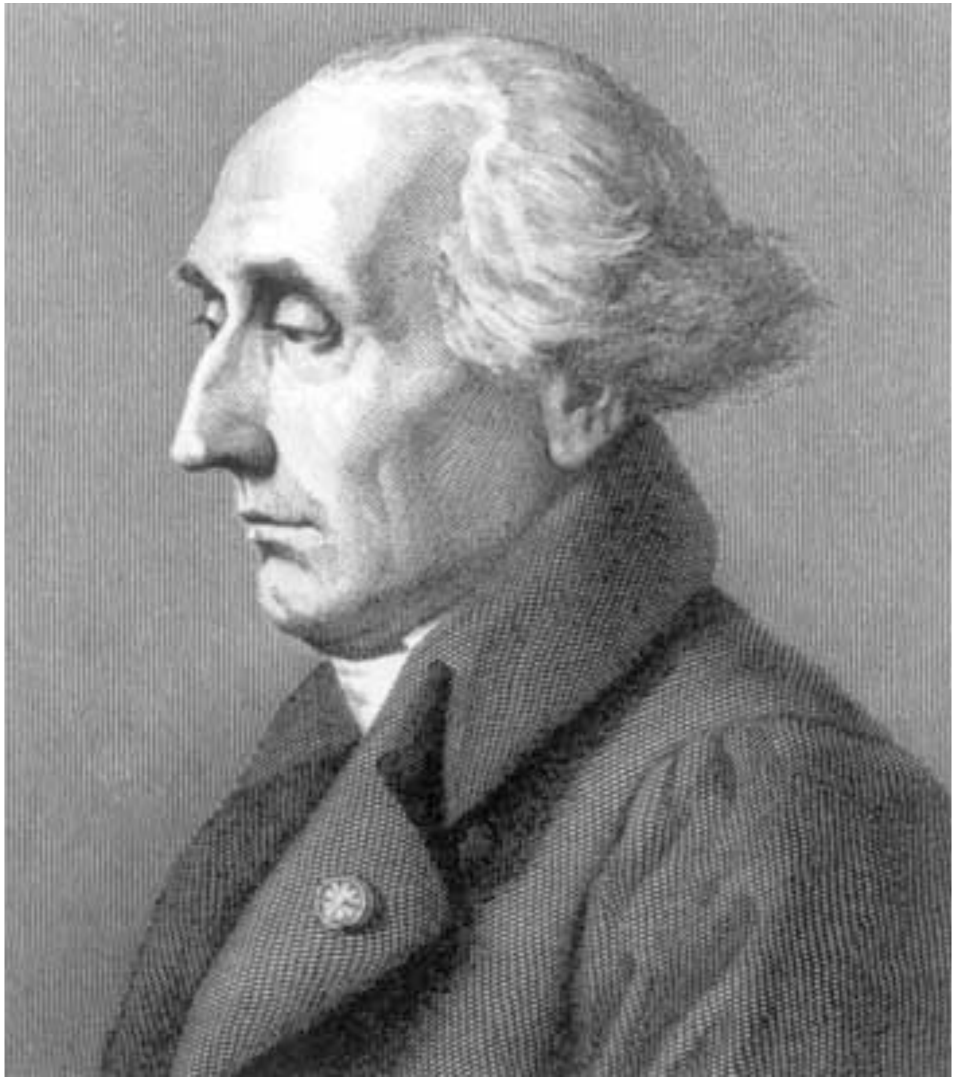
1770 Lagrange

Our first error was committed by Lagrange in 1770



Joseph Louis Lagrange

1736-1813

- Born in Turin, student of Euler
- Replaces Euler in Berlin; later joins the Paris Academy
- During the revolution : metric system, Ecole Normale and Ecole Polytechnique
- Under Napoléon : senator, count of the Empire, grand officer of the Légion d'honneur
- His 'greatest treasure' : his (very) young wife, whom he marries at the age of 56
- Dies in Paris at the age of 77

Lagrange's design problem

Ainsi c'est un problème de maximis et minimis de déterminer la courbe qui, par sa rotation autour de son axe formera une colonne capable de supporter la plus grande charge possible, la hauteur et la masse de la colonne étant données.

Lagrange (1770) *Sur la figure des colonnes*

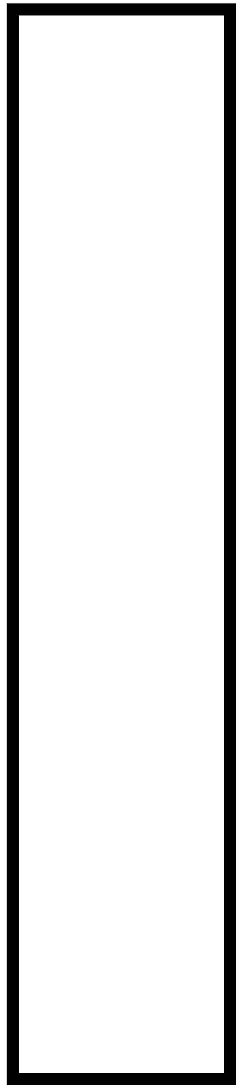
To find the curve which by its revolution determines the column of greatest efficiency.

Truesdell

1770 Lagrange

He proves the following (false) theorem:

The optimal column is a cylinder



the prejudging mistake: acting as if the answer is known, based on incorrect intuition or analogy, or wishful thinking

1783 Legendre

$$\min_{x(\cdot)} \int_a^b L\big(x(t), x'(t)\big)\, dt$$

He proves the following (false) theorem:

If x_* is an extremal satisfying $L_{vv}\big(x(t), x'(t)\big) > 0 \,\forall\, t$, then x_* is a weak local minimum for the problem.

His proof is quite ingenious.

The mistake lies in assuming that a differential equation like $x' = x^2 + 1$ has a solution defined on the interval $[a, b]$.

the existence mistake

Francis Clarke

Functional Analysis, Calculus of Variations and Optimal Control

Functional analysis owes much of its early impetus to problems that arise in the calculus of variations. In turn, the methods developed there have been applied to optimal control, an area that also requires new tools, such as nonsmooth analysis. This self-contained textbook gives a complete course on all these topics. It is written by a leading specialist who is also a noted expositor.

This book provides a thorough introduction to functional analysis and includes many novel elements as well as the standard topics. A short course on nonsmooth analysis and geometry completes the first half of the book whilst the second half concerns the calculus of variations and optimal control. The author provides a comprehensive course on these subjects, from their inception through to the present. A notable feature is the inclusion of recent, unifying developments on regularity, multiplier rules, and the Pontryagin maximum principle, which appear here for the first time in a textbook. Other major themes include existence and Hamilton-Jacobi methods.

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Clarke

Functional Analysis, Calculus of Variations and Optimal Control

Graduate Texts in Mathematics

GTM

Francis Clarke

Functional Analysis, Calculus of Variations and Optimal Control

Springer

1821 Cauchy

In 1821 he proves the following (false) theorem:

$$\boxed{\text{The pointwise limit } f(x) \text{ of a sequence of continuous functions } f_i(x) \text{ is continuous}}$$

Astonishing: Cauchy is the baron of analysis, author of 800 articles, the inventor of epsilon/delta!

the prejudging mistake. But also the inadequate definition mistake

1851 Riemann

He studies solutions u of Laplace's equation

$$u_{xx} + u_{yy} = 0$$

He obtains one by considering

$$\boxed{\begin{array}{l} \min \quad \displaystyle\int_{\Omega} (u_x^2 + u_y^2)\, dx\, dy \\ u = \varphi \text{ (prescribed) on } \partial\Omega \end{array}}$$

which (he says) is evidently attained (Dirichlet principle)

the existence mistake

1889 Poincaré

It is known that two planets constitute a stable system.

In 1887 King Oscar offers a large prize for a solution of the n-body problem.

Poincaré is awarded the prize for showing that the reduced three-body problem is (essentially) stable.

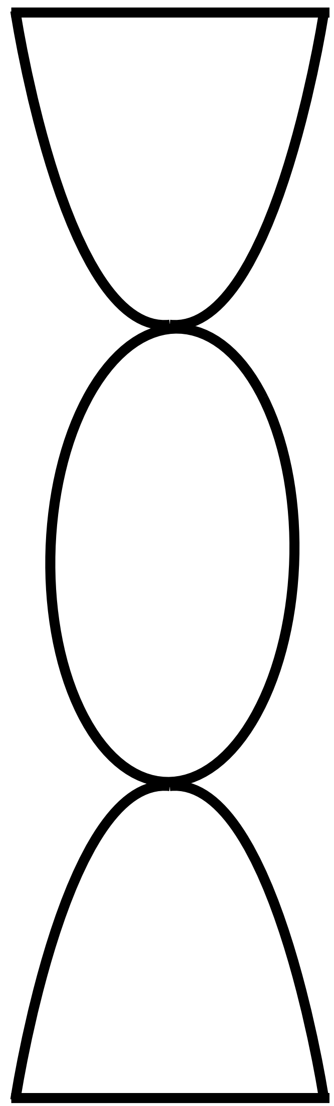
BUT while the prize paper is in proof, serious errors are found. The final paper proves instability (!).

the prejudging mistake

1960 Keller & Tadjbaksh

They prove the following (false) theorem:

The optimal column
has zero width
at two points



the smoothness mistake

1968 Arrow

The necessary conditions are known as the Maximum Principle

$$\min \; \ell(x(b)) \;\text{ where}$$

$$\begin{cases} \text{state } x(\cdot) \text{ and control } u(\cdot) \\ \text{defined on } [a,b] \\ \quad x'(t) = f(x(t), u(t)) \text{ a.e.} \\ \quad u(t) \in U \text{ a.e.} \\ (x(a), x(b)) \in S \end{cases}$$

The Nobel-winning economist proves, by economic reasoning, a much-cited version of the Maximum Principle for problems in which U = U(x).

The conclusions are identical to the usual ones for the case in which U does NOT depend on x.

the prejudging mistake

1976 Clarke

$$\min\ \ell(x(b)) \ \text{where}$$

$$\begin{cases} x'(t) \in F(t, x(t)) \ \text{a.e.} \\ (x(a), x(b)) \in S \end{cases}$$

differential
inclusion

He derives (incorrectly) the Hamiltonian inclusion necessary conditions.

the prejudging mistake

(corrected in 2005)

Lots of people, often, today

The Deductive Method:

1. Prove (a priori) that a solution exists
2. Apply correct necessary conditions
3. Identify (through elimination and comparison) a unique candidate.

$\Rightarrow$ the candidate solves the problem

It is a fallacy to take existence for granted (to omit step 1) in applying the deductive method

This fallacy is especially common in optimal control.

Occasionally, the class of controls is not even specified, or taken to be PWC (prejudge the answer)...

Sometimes, existence is justified by the fact that a "real" phenomenon is being modeled.

In logic, this is called the "fallacy of misplaced concreteness".

For optimal control problems, a correct analysis would involve one of:

- existence theory (measurable controls)
- convexity
- relaxed controls (measures)
- verification functions (nonsmooth)

(beyond the comfort zone of many engineers and economists)

A major source of confusion is the dynamic programming approach (1950's and 60's): everything is assumed smooth, continuous, etc.

This leads to a misleading climate: the ultimate smoothness mistake!

Consider control systems that are GAC.

GAC		stable
$x'(t) = f(x(t), u(t))$ a.e. $u(t) \in U$ a.e.	$\Longleftrightarrow$	$x'(t) = g(x(t))$ a.e.

Fact:

stable

$x'(t) = g(x(t))$ a.e.

$\Longleftrightarrow$

there is a smooth Lyapunov function

But:

GAC

$x'(t) = f(x(t), u(t))$ a.e.

$u(t) \in U$ a.e.

$\Longleftrightarrow$

there is a control Lyapunov function (nonsmooth in general)

Let us consider the issue of stabilizing feedbacks.

GAC

$$x'(t) = f(x(t), u(t)) \text{ a.e.}$$
$$u(t) \in U \text{ a.e.}$$

Goal:

find $k(x)$ with values in U such that

$$x'(t) = g(x(t)) := f(x(t), k(x(t)))$$

is stable.

It is a fallacy to assume that k can be taken to be continuous. (Example: NHI)

Question: What does it mean to put a **discontinuous** function $k(x)$ into f?

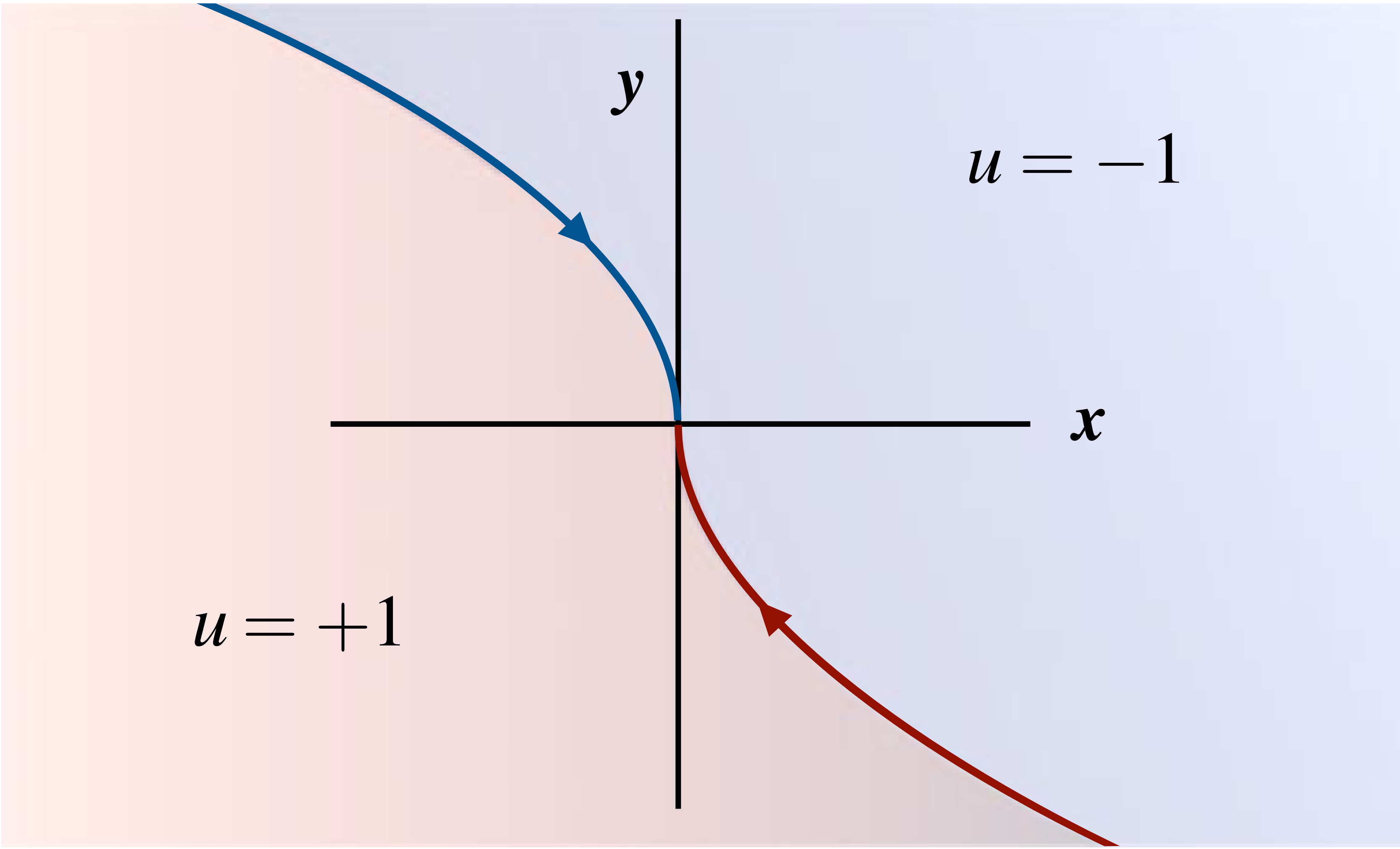
Or: to consider solutions of $x' = g(x)$ when g is discontinuous?

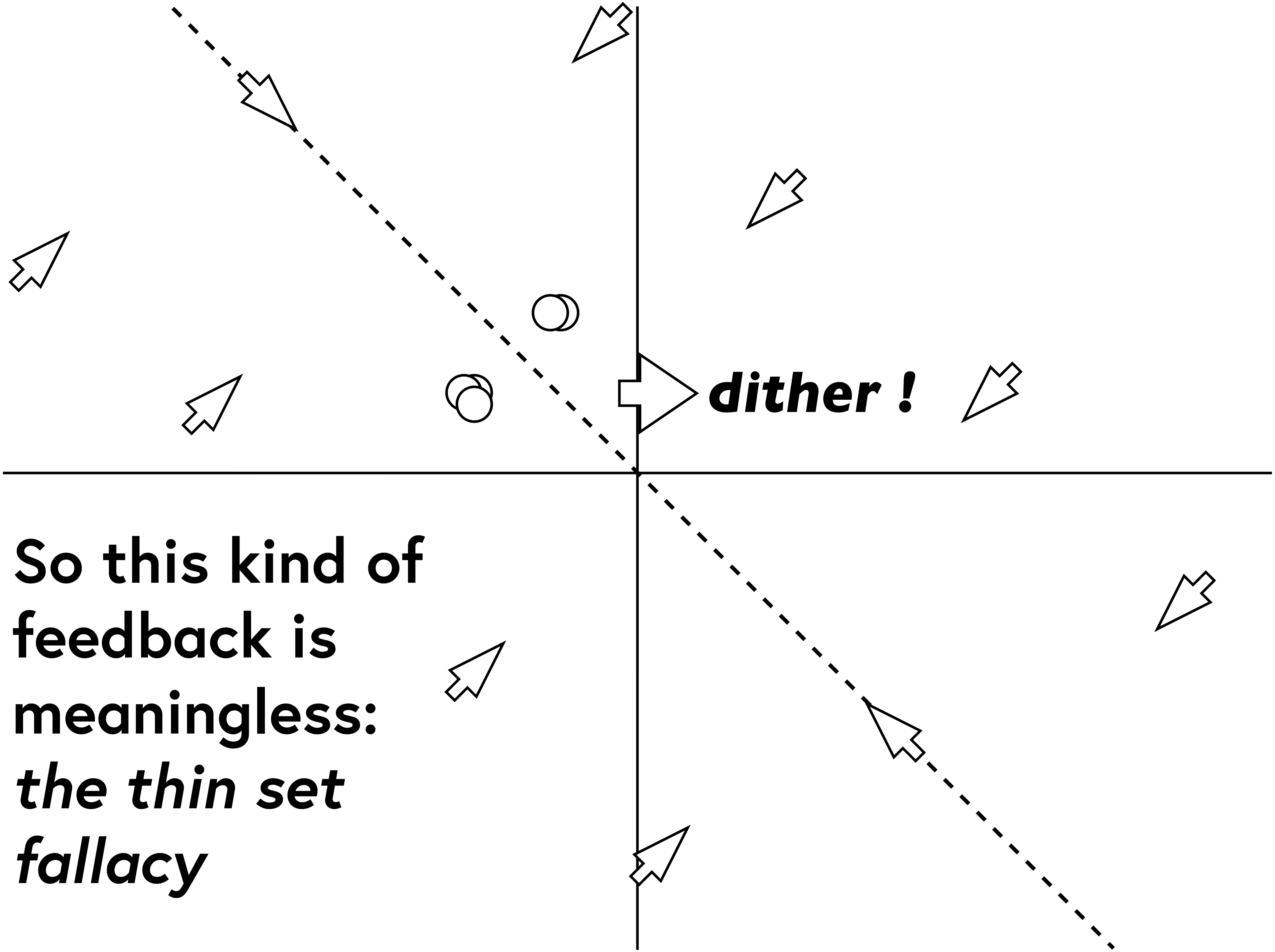
There is a longstanding inadequate definition error in considering discontinuous feedbacks

Example: minimal time for the double integrator

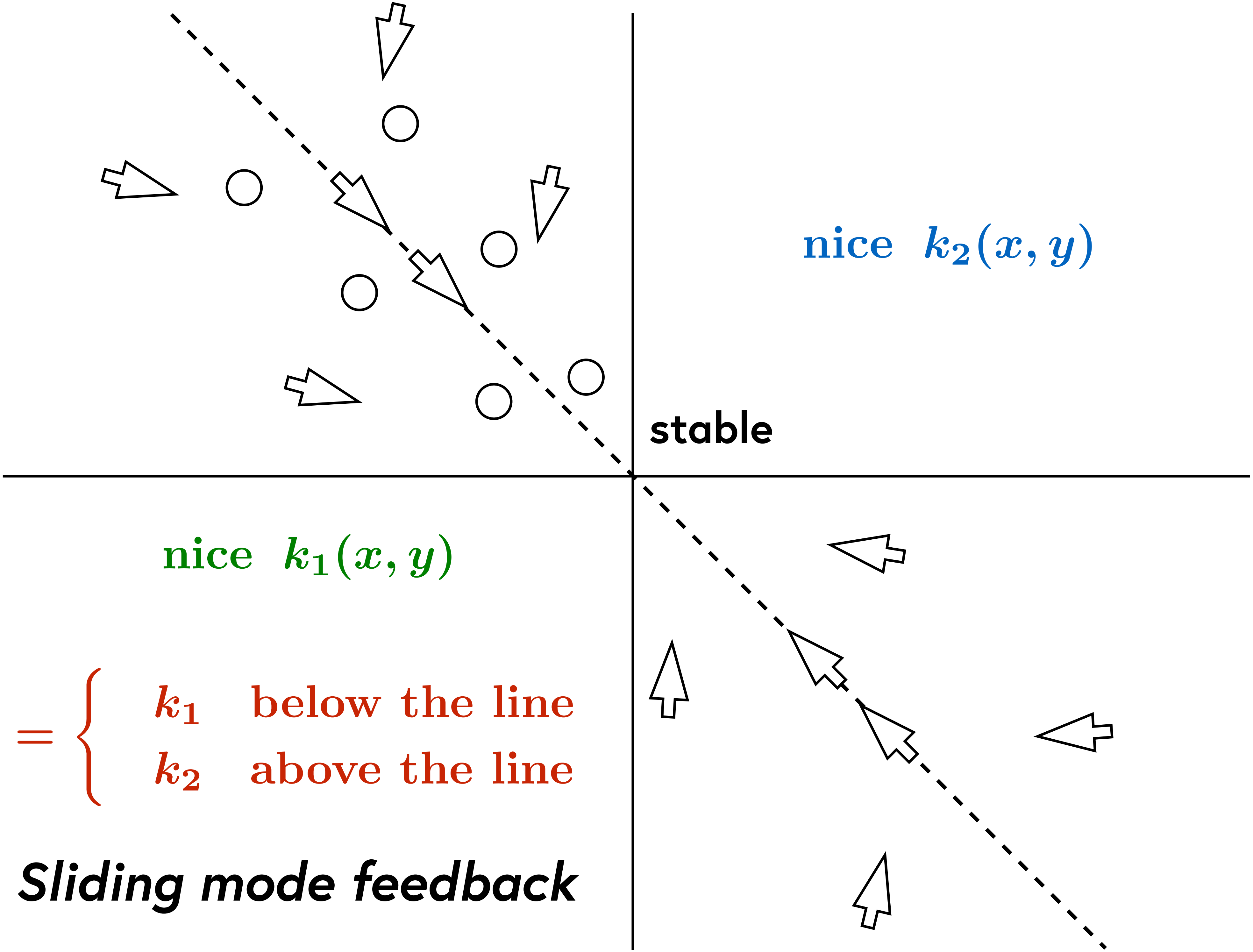
$$x'' = u(t) \in [-1, 1] \text{ a.e.} \qquad \begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$\min T : x(T) = y(T) = 0$$





So this kind of feedback is meaningless:
the thin set fallacy



$$k = \begin{cases} k_1 & \text{below the line} \\ k_2 & \text{above the line} \end{cases}$$

Sliding mode feedback

progress has been made, notably:

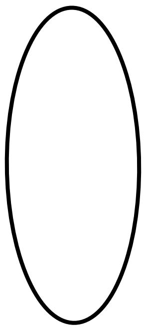
Theorem (Clarke, Ledyaev, Sontag, Subbotin 1997)

A system is GAC if and only if it is stabilizable by feedback (in the sample-and-hold sense).

A rigorous approach to sliding feedback control (Clarke & Vinter 2009)

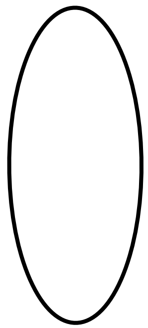
There is a parallel universe of alternative facts out there. You must choose:

The blue pill

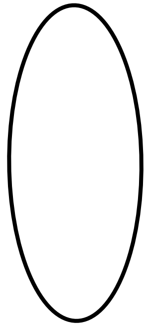


or

The red pill



The blue pill



Minima are always attained

Necessary conditions always apply

Equations have global smooth solutions

Unique extremals must be solutions

Intuition is always right

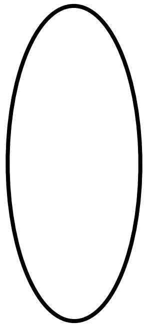
Dependences and value functions are smooth

Controllable systems admit continuous stabilizing feedbacks and smooth control Lyapunov functions

Nonlinear control systems are always linearizable

The values of a feedback on a set of measure zero can effectively steer a system

The red pill



"You take the red pill: ... all I'm offering is the truth. Nothing more." **(Morpheus)**

Welcome to the real world

It may be tempting to remain in the world of alternative facts:

"Why oh why didn't I take the blue pill?" **(Cypher)**

But we must resist, and learn from our errors!

"Truth emerges more readily from error than from confusion" **(Francis Bacon 1611)**