Moreau's sweeping process and its control

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OPTIMOS

ANNOS!

The dynamics of the sweeping process

Consider a moving set $C(t)$, depending on the time $t \in [0, T]$, and an initial condition $x_0 \in C(0)$.
In several contexts, the modelization of the displacement $x(t)$ of the initial condition x_0 subject to the dragging, or sweeping due to the displacement of $C(t)$ pops up.
It is natural to think that the point $x(t)$ remains at rest until it is caught by the boundary of $C(t)$ and then its velocity is normal to $\partial C(t)$.
It is a kind of one sided movement.

Formally, the sweeping process (*processus de rafle*) is the rate independent differential inclusion with initial condition

$$\dot{x}(t) \in -N_{C(t)}(x(t)), \quad x(0) = x_0 \in C(0).$$

Here $N_C(x)$ denotes the normal cone to C at $x \in C$. In particular,

$$\begin{array}{ll} N_C(x) = \{0\} & \text{if } x \in \mathrm{intC} \\ N_C(x) = \emptyset & \text{if } x \notin C. \end{array}$$

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The simplest example is the play operator:

$$\dot{x}(t) \in -N_{C+u(t)}(x(t)),$$

namely $C(t)$ is a translation.

Classical assumptions are:

- $x \in H$, a Hilbert space
- $t \mapsto C(t)$ is Lipschitz continuous
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- The *catching up* algorithm

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There are also many generalizations.

The perturbed sweeping process, in the simpler, yet relevant, case where C is constant, namely

$$\dot{x}(t) \in -N_C(x(t)) + f(x(t)), \quad x(0) = x_0 \in C(0), \tag{1}$$

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If C is constant and mildly nonconvex, (1) is equivalent to

$$\dot{x} \in \mathrm{proj}_{T_C(x)}(f(x)).$$

Control problems

Consider now the dynamics

$$\dot{x}(t) \in -N_{C(t)}(x(t)) + f(x(t), u), \; u \in U.$$

The control may appear in the colored items: shape optimization and classical control.

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but before that just a statement of the H-J characterization of a value function. The time dependent case can be treated as well, but for simplicity I will treat mostly the case $C(t) \equiv C$.

Consider the problem

$$(SP) \quad \begin{cases} \text{Minimize } T \text{ over solutions of} \\ \dot{x}(t) \in -N_{C(t)}(x(t)) + G(x(t)) \text{ a.e.} \\ x(t_0) = x_0 \in C(t_0), \quad x(T) \in S. \end{cases}$$

Assumptions: the moving set $C(.)$ is Lipschitz, closed and convex valued; $G(.)$ is Lipschitz and compact and convex valued.

Compatibility Condition: $\exists\ \bar{t} > 0$ such that $C(\bar{t}) \cap S \neq \emptyset$.

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Method: characterization through invariance of epi/hypograph.

HJ inequalities for SP (C.-Palladino; '15)

Define:

$$H_-(\tau, x, \lambda, p) := \min_{v\in\{0\}\times\{-N_{C(\tau)}(x)\cap(L_C+M_G)\mathbb{B}\}\times\{0\}} v\cdot p + \min_{v\in\{1\}\times\{G(x)\}\times\{-1\}} v\cdot p,$$

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Theorem: Assume $T(.,.)$ continuous. Then $T(.,.)$ is the unique function satisfying:

$$T(t,x) > 0 \qquad \forall\,(t,x) \in \mathrm{Gr}\, C \qquad \text{for which} \quad x \notin S,$$

$$T(t,x) = 0 \qquad \forall\,(t,x) \in \mathrm{Gr}\, C \qquad \text{for which} \quad x \in S,$$

$$H_-(t,x,T(t,x),p) \le 0 \;\; \forall (t,x) \in \mathrm{Gr}\, C, x \notin S, \;\; \forall p \in N^P_{\mathrm{epi}\, T}(t,x,T(t,x)),$$
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Remarks. 1) If $x\in \mathrm{int}\,C(t)$ and $x\notin S$, then the classical *HJ* equality holds. 2) The structure of the Hamiltonians: observe the min in the normal part.

A maximum principle for the controlled sweeping process (Arroud-C., 2016 and 2017, Palladino-C., 2017+)

Minimize $h(x(T))$ subject to

$$\begin{cases} \dot{x}(t) \in -N_C(x(t)) + f(x(t), u(t))), \\ x(0) = x_0 \in C(0) , \end{cases}$$

with respect to $u : [0, T] \to U$, u measurable (C is smooth and convex, with $0 \in C$).

Method: Moreau-Yosida approximation (inspired by Brokate & Kreičí, DCDS-B 2013).

Given $\varepsilon > 0$ and a global minimizer $(\bar{x}, \bar{u})$, one considers the problem of minimizing

$$h(x(T)) + \frac{1}{2}\int_0^T |u - \bar{u}|^2 \, dt$$

subject to

$$\dot{x}_\varepsilon(t) = -\frac{1}{\varepsilon}\big(x_\varepsilon(t) - \mathrm{proj}_C(x_\varepsilon(t))\big) + f(x_\varepsilon(t), u), \quad x(0) = x_0$$

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It is possible to prove:
Existence and uniqueness of a solution and the estimate

$$\|d(x_\varepsilon(t), C)\|_\infty \leq \mathrm{const}\, \varepsilon.$$

The strong L^2 convergence of a sequence of minimizers u_ε to $\bar{u}$.

This is a classical control problem:
– write down necessary conditions.
– try to pass to the limit along the adjoint vectors and PMP.

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Remark. $x_\varepsilon(t) - \mathrm{proj}_C(x_\varepsilon(t)) = \nabla_x \frac{1}{2} d^2(x_\varepsilon(t), C)$ (thus $\nabla_x d^2(., C)$ is Lipschitz).

Then necessary conditions read as

$$\begin{cases} -\dot{p}_\varepsilon(t) & = \left(\frac{-1}{2\varepsilon}\nabla_x^2 d^2(x_\varepsilon(t), C) + \nabla_x f(x_\varepsilon(t), u_\varepsilon(t))\right) p_\varepsilon(t),\ t \in [0, T] \\ -p_\varepsilon(T) & = \nabla h(x_\varepsilon(T)) \end{cases}$$

$+$ PMP.

Computation (valid for all $x \notin \partial C$)

$$\underbrace{\nabla_x^2 d^2(x, C) = d(x, C)\nabla_x^2 d(x, C)}_{\text{continuous}} + \underbrace{\nabla_x d(x, C) \otimes \nabla_x d(x, C)}_{\text{discontinuous at } \partial C}$$

$$\frac{1}{\varepsilon}\nabla_{x_\varepsilon}^2 d^2(x_\varepsilon, C) = \underbrace{\frac{d(x_\varepsilon, C)}{\varepsilon}\nabla_x^2 d(x_\varepsilon, C)}_{\text{bounded}} + \underbrace{\frac{1}{\varepsilon}\nabla_x d(x_\varepsilon, C) \otimes \nabla_x d(x_\varepsilon, C)}_{??}$$

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continuous discontinuous at ∂C

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bounded ??

Brokate & Kreičí smooth out the distance. (disegna) This simplifies the estimate of the red part, while complicates the estimate of the blue part: they need to assume **uniform strict convexity** and **time independence** of C.

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We consider a weaker penalization: replace

$$\dot{x}_\varepsilon = -\frac{1}{2\varepsilon}\nabla d^2(x_\varepsilon, C) + f(x_\varepsilon(t), u)$$

by

$$\dot{x}_\varepsilon = -\frac{1}{3\varepsilon}\nabla d^3(x_\varepsilon, C) + f(x_\varepsilon(t), u).$$

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Actually $d^3(\cdot, C)$ is of class $\mathcal{C}^2(\mathbb{R}^n)$. Since the penalization is weaker, proving the convergence of x_ε is more difficult: the estimate is now

$$\|d(x_\varepsilon, C)\|_{L^2} \sim \sqrt{\varepsilon}$$

in place of

$$\|d(x_\varepsilon, C)\|_{L^\infty} \sim \varepsilon$$

and here the time independence of C is important.

Now necessary conditions read as

$$\begin{cases} -\dot{p}_\varepsilon & = \left(\frac{-1}{3\varepsilon}\nabla_x^2 d^3(x_\varepsilon, C) + \nabla_x f(x_\varepsilon, u_\varepsilon)\right)p_\varepsilon \\ -p_\varepsilon(T) & = \nabla h(x_\varepsilon(T)) \end{cases}$$

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$+$ PMP, where

$$\frac{1}{\varepsilon}\nabla_x^2 d^3(x_\varepsilon, C(t)) =$$

$$= \frac{3d^2(x_\varepsilon, C)}{\varepsilon}\nabla_x^2 d(x_\varepsilon, C) + \frac{6d(x_\varepsilon, C)}{\varepsilon}\nabla_x d(x_\varepsilon, C) \otimes \nabla_x d(x_\varepsilon, C).$$

The bad part is now easier to handle.

Alternatively, in order to treat the time dependent case one needs to go back to the classical Moreau-Yosida approximation and add an assumption on the optimal trajectory that prevents all optimal trajectories of the approximate problems from spending too much time on $\partial C(t)$. This is a kind of outward pointing condition that I will explain later.

Theorem (Arroud-C, 2016 and 2017). Let (x_*, u_*) be a global minimizer satisfying the outward (or inward) pointing condition. Then there exist a BV adjoint vector $p : [0, T] \to \mathbb{R}^n$, a finite signed Radon measure μ on $[0, T]$, and measurable vectors $\xi, \eta : [0, T] \to \mathbb{R}^n$, with $\xi(t) \geq 0$ for μ-a.e. t and $0 \leq \eta(t) \leq \beta + \gamma$ for a.e. t, satisfying the following properties:

- (adjoint equation) for all continuous functions $\varphi : [0, T] \to \mathbb{R}^n$

$$
\begin{aligned}
-\int_{[0,T]} \langle \varphi(t), dp(t) \rangle = & -\int_{[0,T]} \langle \varphi(t), \nabla_x d(x_*(t), C(t)) \rangle \xi(t)\, d\mu(t) \\
& -\int_{[0,T]} \langle \varphi(t), \nabla_x^2 d(x_*(t), C(t)) p(t) \rangle \eta(t)\, dt \\
& +\int_{[0,T]} \langle \varphi(t), \nabla_x f(x_*(t), u_*(t)) p(t) \rangle\, dt,
\end{aligned}
$$

- (transversality condition) $\quad -p(T) = \nabla h(x_*(T))$,
- (maximality condition)

$$\langle p(t), \nabla_u f(x_*(t), u_*(t)) u_*(t) \rangle = \max_{u \in U} \langle p(t), \nabla_u f(x_*(t), u_*(t)) u \rangle \text{ for a.e. } t.$$

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The adjoint vector is *absolutely continuous* in the *interior* of the interval where $\bar{x}(t) \in \partial C$.

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A closer look at an example.

An example: in $\mathbb{R}^2$, minimize $x(1) + y(1)$ subject to

$$(\dot{x}, \dot{y}) \in -N_C(x, y) + (u_x, u_y), \qquad |u_x|, |u_y| \leq 1$$

$$\left(\text{or } |u_x| \leq 1,\ -1 \leq u_y \leq -\frac{1}{2}\right)$$

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Some degeneracy of the y-component of the adjoint vector is to be expected (is not a pathology).

The adjoint vector p^x is absolutely continuous on $(0,1]$, $p^y)$ is absolutely continuous on $(0,1)$ on $(0,\bar{t}) \cup (\bar{t},1]$ $\dot{p}^x = 0$, $\dot{p}^y = 0$ a.e. on $[0,T]$, $p^x(1) = p^y(1) = -1$, and $p^y(1-)+1 = 1$, namely $p^y(1-) = 0$. Thus the adjoint vector (p^x,p^y) is :

$$\begin{aligned} p^x(t) &= -1 \quad \text{for all } t \in [0,1] \\ p^y(t) &= 0 \quad\ \ \text{for all } t \in [0,1) \\ p^y(1) &= -1 \quad \Rightarrow \mu = -\delta_1. \end{aligned}$$

The maximum condition reads as

$$\langle(-1,0),(\bar{u}^x,\bar{u}^y)\rangle = \max_{|u_x|,|u_y|\le 1}\langle(-1,0),(u_x,u_y)\rangle \quad \text{for } 0\le t<1,$$

i.e.,

$$-\bar{u}^x = \max_{|u_x|\le 1}\{-u_x\} \Rightarrow \bar{u}^x \equiv 1, \qquad \text{BUT no information on } \bar{u}^y.$$

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Work in progress (with Michele Palladino) to find another method to obtain better necessary optimality conditions.

Main ideas:

- we can treat only the case in which the optimal trajectory touches the boundary of C **on time intervals**;
- in the time interval where $\bar{x}(t) \in \partial C$, we treat separately the times t in which $f(\bar{x}(t), \bar{u}(t)) \cdot \nabla g(\bar{x}(t)) \leq 0$ (i.e., the normal cone is not active) or $f(\bar{x}(t), \bar{u}(t)) \cdot \nabla g(\bar{x}(t)) > 0$ (i.e., the normal cone is active);
- in the first case, we treat the problem as a traditional state constrained optimal control problem (we add the state constraint $x(t) \in C$);
- this allows a further discontinuity in the adjoint vector, that prevents its degeneracy when the constraint is not active;
- a better auxiliary problem (using classical tricks in necessary optimality conditions) allows to recover the classical PMP also for local minimizers.

Problems still to be solved:

- we can't treat the general case, where $\bar{x}(t)$ may touch ∂C on an arbitrary measurable set;
- we need the scalar product $f(\bar{x}(t), \bar{u}(t)) \cdot \nabla g(\bar{x}(t))$ to be bounded away from zero, uniformly with respect to t.

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Colombo, Palladino
hopefully converging soon

THANK YOU FOR YOUR ATTENTION.