Small-time global controllability of the Navier-Stokes equation with the Navier slip boundary conditions

Jean-Michel Coron

UPMC (LJLL) and ETH-ITS

New trends in Control Theory and PDEs

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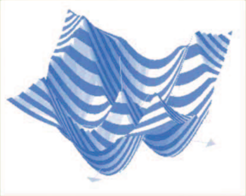
On the occasion of the 60th birthday of Piermarco Cannarsa

Progress in Nonlinear Differential Equations and Their Applications

Piermarco Cannarsa
Carlo Sinestrari

Semiconcave Functions, Hamilton–Jacobi Equations, and Optimal Control



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PERSISTENT REGIONAL NULL CONTROLLABILITY FOR A CLASS OF DEGENERATE PARABOLIC EQUATIONS

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CARLEMAN ESTIMATES FOR A CLASS OF DEGENERATE PARABOLIC OPERATORS*

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Abstract. Given $\alpha \in [0,2)$ and $f \in L^2((0,T) \times (0,1))$, we derive new Carleman estimates for the degenerate parabolic problem $w_t + (x^\alpha w_x)_x = f$, where $(t,x) \in (0,T) \times (0,1)$, associated to the boundary conditions $w(t,1) = 0$ and $w(t,0) = 0$ if $0 \leq \alpha < 1$ or $(x^\alpha w_x)(t,0) = 0$ if $1 \leq \alpha < 2$. The proof is based on the choice of suitable weighted functions and Hardy-type inequalities. As a consequence, for all $0 \leq \alpha < 2$ and $\omega \subset\subset (0,1)$, we deduce null controllability results for the degenerate one-dimensional heat equation $u_t - (x^\alpha u_x)_x = h\chi_\omega$ with the same boundary conditions as above.

The Navier Stokes equations

Let Ω be a smooth bounded non empty open subset of $\mathbb{R}^d$, $d \in \{2, 3\}$. We are interested in the Navier-Stokes equations

$$(1) \qquad \left\{ \begin{array}{l} y_t - \Delta y + (y \cdot \nabla)\, y + \nabla p = 0,\ t \in [0, T], x \in \overline{\Omega}, \\ \text{div } y = 0,\ t \in [0, T], x \in \overline{\Omega}, \end{array} \right.$$

where, at time $t \in [0, T]$ and at the position $x \in \overline{\Omega}$, $y(t, x) \in \mathbb{R}^d$ is the velocity of the viscous incompressible fluid. We assume that we are able to prescribe y on a non empty open subset Γ of $\partial\Omega$.

The Navier slip boundary condition

The Navier slip boundary conditions are

$$(1) \qquad y \cdot n = 0 \quad \text{and} \quad [D(y)n + Ay]_{\tan} = 0 \text{ on } \partial\Omega \setminus \Gamma.$$

Here and in the sequel, n denotes the outward normal to $\partial\Omega$. For a vector field f, we introduce $[f]_{\tan}$ its tangential part and $D(f)$ the rate of strain tensor (or shear stress) which are defined by:

$$(2) \qquad [f]_{\tan} := f - (f \cdot n)n, \; D_{ij}(f) := \frac{1}{2}\left(f^j_{x_i} + f^i_{x_j}\right).$$

Eventually, in (1), A is a smooth matrix valued function on $\partial\Omega$, describing the friction near the boundary. This is a generalization of the usual condition involving a single scalar parameter $\alpha \geq 0$ (i.e. $A = \alpha I_d$). For flat boundaries, such a scalar coefficient measures the amount of friction. When $\alpha = 0$ and the boundary is flat, the fluid slips along the boundary without friction and there is no boundary layers. When $\alpha \to +\infty$, the friction is so intense that the fluid is almost at rest near the boundary; condition (1) converges to the Dirichlet condition.

The controllability problem

The question of small time global exact null controllability asks whether, for any $T > 0$ and any initial data y^0 (in some appropriate space), there exists a trajectory y defined on $[0, T] \times \Omega$, which is a solution to

$$(1) \qquad \begin{cases} y_t + (y \cdot \nabla)\, y - \Delta y + \nabla p = 0 \text{ in } (0,T) \times \Omega \\ \ \text{div } y = 0, \\ y \cdot n = 0 \text{ and } \quad [D(y)n + Ay]_{\tan} = 0 \text{ on } (0,T) \times (\partial\Omega \setminus \Gamma), \end{cases}$$

satisfying $y(0, \cdot) = y^0$ and $y(T, \cdot) = 0$. In this formulation, we see system (1) as an under-determined system. The controls used are the (implicit) boundary conditions on Γ and can be recovered from the constructed trajectory y itself. This problem was suggested by Jacques-Louis Lions in the late 80's (for the Dirichlet boundary condition, also called the no slip Stokes condition: $y = 0$ on $\partial\Omega \setminus \Gamma$).

The mathematical framework

We define the space $L^2_\sigma(\Omega)$ as the closure in $L^2(\Omega)$ of the space of smooth divergence free vector fields which are tangent to $\partial\Omega \setminus \Gamma$. For $T > 0$ and $y^0 \in L^2_\sigma(\Omega)$, we say that $y \in \mathcal{C}^0_w([0,T]; L^2_\sigma(\Omega)) \cap L^2((0,T); H^1(\Omega))$ is a weak Leray solution to our Navier-Stokes equation with initial data y^0 when

$$
\begin{aligned}
(1) \quad -\int_0^T \int_\Omega y \cdot \psi_t + \int_0^T \int_\Omega y \cdot \nabla y \cdot \psi + 2 \int_0^T \int_\Omega D(y) : D(\psi) \\
- 2 \int_0^T \int_{\partial\Omega\setminus\Gamma} [Ay]_{\tan} \cdot \psi = \int_\Omega y^0 \cdot \psi(0,\cdot),
\end{aligned}
$$

for any $\psi \in \mathcal{C}^\infty([0,T] \times \bar{\Omega})$ which is divergence free, tangent to $\partial\Omega$, vanishes at $t = T$ and vanishes on the controlled boundary Γ.

Main result

Theorem (JMC, F. Marbach and F. Sueur (2016))

Assume that Γ *is an open subset of* $\partial\Omega$ *which meets every connected component of* $\partial\Omega$. *Let* $T > 0$ *and* $y^0 \in L^2_\sigma(\Omega)$. *There exists a weak Leray solution*

$$y \in \mathcal{C}^0_w([0,T]; L^2_\sigma(\Omega)) \cap L^2((0,T); H^1(\Omega))$$

satisfying $y(0,\cdot) = y^0$ *and* $y(T,\cdot) = 0$.

Known results and previous works: local results

A first approach to study the controllability of Navier-Stokes is to see the quadratic term as a perturbation term and obtain results using mostly the Laplacian. This kind of approach is efficient only for local results, where the quadratic term is indeed small. It is not clear that it is a good approach to get global controllability.

For the Dirichlet boundary condition, O. Imanuvilov in 2001 and E. Fernández-Cara, S. Guerrero, O. Imanuvilov and J.-P. Puel in 2004 proved small time local null controllability. Their proof uses Carleman estimates.

For Navier boundary conditions, Havârneanu, Popa and Sritharan proved in 2006 a local controllability result in 2D domains. In 2006, Guerrero proved the small time local null controllability for 2D and 3D domains, with general (non-linear) Navier-type boundary conditions.

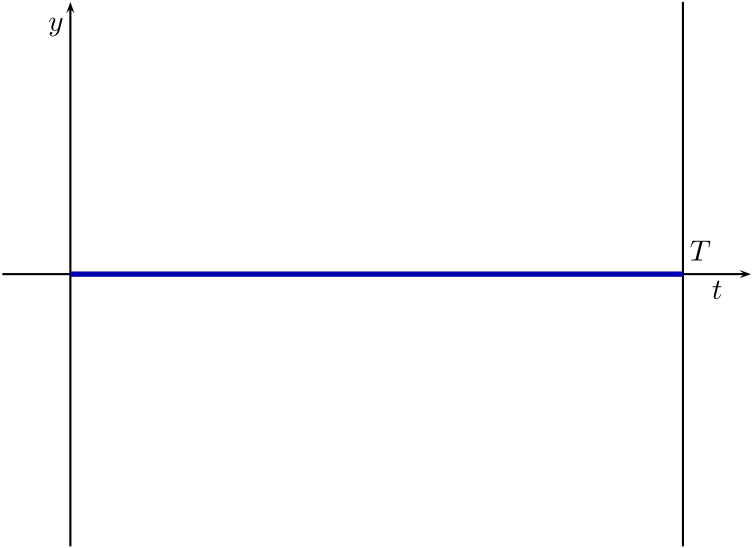
Known results and previous works: global results

For Navier boundary conditions in 2D, JMC proved (1996) a small time global approximate null controllability result. More precisely a good approximate controllability can be obtained in the interior of the domain. However, this is not the case near the boundaries. The approximate controllability is obtained in the space $W^{-1,\infty}$, which is not a strong enough space to be able to conclude to global exact null controllability using a local result.

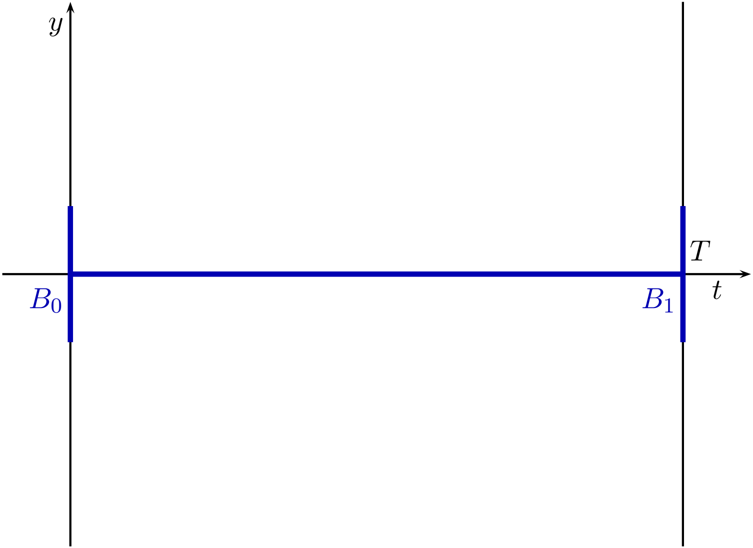
The global null controllability in small time has been proved when $\Gamma = \partial\Omega$ by JMC and A. Fursikov (1996) in dimension 2 and by A. Fursikov and O. Imanuvilov (1999) in dimension 3. The proof is based on the return method (JMC (1992)).

Return method and global controllability

Return method and global controllability



Return method and global controllability



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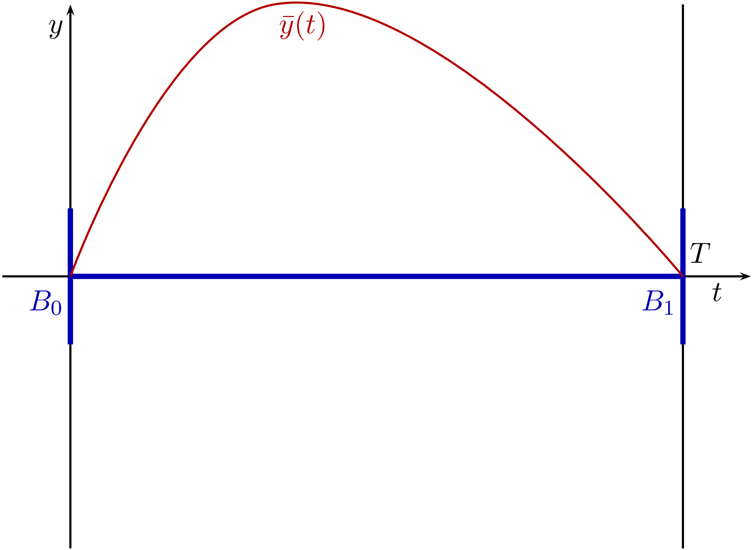
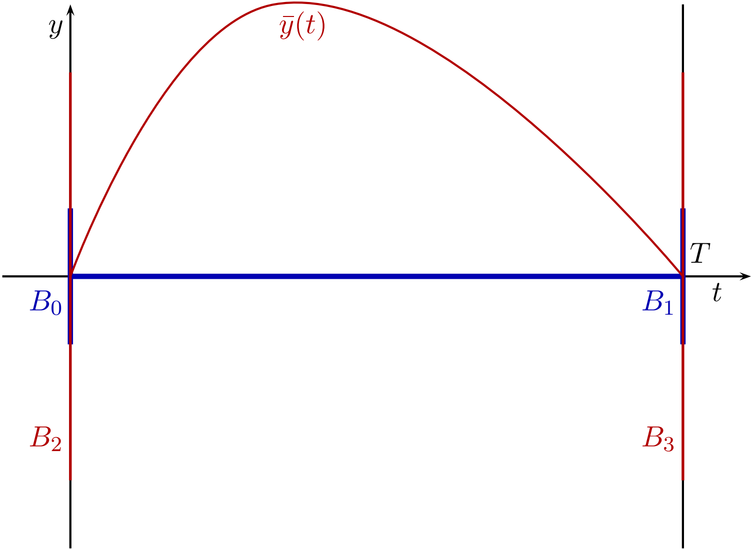
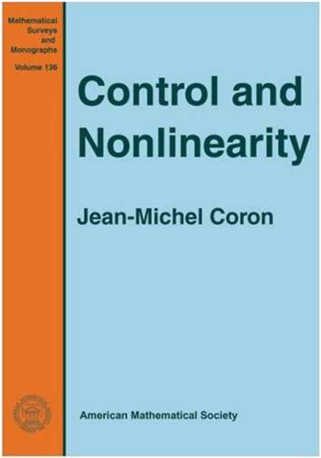


Return method and global controllability



Return method: Commercial break



JMC, Control and nonlinearity, Mathematical Surveys and Monographs, 136, 2007, 427 pp.

The two key terms of the Navier-Stokes equation

The Navier-Stokes equation

$$(1) \qquad \begin{cases} y_t - \Delta y + (y \cdot \nabla)\, y + \nabla p = 0,\ t \in [0,T], x \in \overline{\Omega}, \\ \text{div } y = 0,\ t \in [0,T], x \in \overline{\Omega}, \end{cases}$$

has two main terms

- $-\Delta y$, which is linear,
- $(y \cdot \nabla)\, y$, which is quadratic.

Note that if we omit the linear term, one gets

$$(2) \qquad \begin{cases} y_t + (y \cdot \nabla)\, y + \nabla p = 0,\ t \in [0,T], x \in \overline{\Omega}, \\ \text{div } y = 0,\ t \in [0,T], x \in \overline{\Omega}, \end{cases}$$

which is the Euler equation of incompressible fluids. Let us emphasize that the boundary conditions are on $(0,T) \times (\partial\Omega \setminus \Gamma)$

- For the Euler equations: $y \cdot n = 0$,
- For the Navier-Stokes equations with the Navier slip condition $y \cdot n = 0$ and $[D(y)n + Ay]_{\tan} = 0$

An example in finite dimension

We consider the following control system

$$\dot{y} = Ay + F(y) + Bu(t), \tag{1}$$

where the state is $y \in \mathbb{R}^n$, the control is $u \in \mathbb{R}^m$, A is a $n \times n$ matrix, B is a $n \times m$ matrix and $F \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ is quadratic: $F(\lambda y) = \lambda^2 F(y)$, $\forall \lambda \in [0, +\infty)$, $\forall y \in \mathbb{R}^n$. For the "application" to incompressible fluids (1) is the Navier-Stokes control system, while

$$\dot{y} = F(y) + Bu(t), \tag{2}$$

where the state is $y \in \mathbb{R}^n$, the control is $u \in \mathbb{R}^m$ is the Euler control system.

The main assumption

Main assumption

We assume that there exists a trajectory $(\bar{y}, \bar{u}) \in C^0([0, T_0]; \mathbb{R}^n) \times L^\infty((0, T_0); \mathbb{R}^m)$ *of the control system* $\dot{y} = F(y) + Bu(t)$ *such that the linearized control system around* $(\bar{y}, \bar{u})$ *is controllable and such that* $\bar{y}(0) = \bar{y}(T_0) = 0$.

Remark

The controllability of

(3) $$\dot{y} = F(y) + Bu$$

is equivalent to the existence of trajectory $(\bar{y}, \bar{u}) \in C^0([0, T_0]; \mathbb{R}^n) \times L^\infty((0, T_0); \mathbb{R}^m)$ *of the control system* (3) *such that the linearized control system around* $(\bar{y}, \bar{u})$ *is controllable and such that* $\bar{y}(0) = \bar{y}(T_0) = 0$ *(JMC (1992, 1994)).*

Remark

One has $F(0) = 0$. *Hence* $(0,0)$ *is an equilibrium of the control system* $\dot{y} = F(y) + Bu(t)$. *The linearized control system around this equilibrium is* $\dot{y} = Bu$, *which is not controllable if (and only if)* B *is not onto.*

One has the following theorem.

Theorem

Under the above assumptions, the control system (1) *is globally controllable in arbitrary time: For every* $T > 0$, *for every* $y^0 \in \mathbb{R}^n$ *and for every* $y^1 \in \mathbb{R}^n$, *there exists* $u \in L^\infty((0,T);\mathbb{R}^m)$ *such that*

$$(\dot{y} = Ay + F(y) + Bu(t),\, y(0) = y^0) \Rightarrow (y(T) = y^1)\,. \tag{1}$$

Proof of the theorem

Let $y^0 \in \mathbb{R}^n$ and $y^1 \in \mathbb{R}^n$. Let

$$\begin{array}{lccc} G: & \mathbb{R} \times L^\infty((0,T_0);\mathbb{R}^m) & \to & \mathbb{R}^n \\ & (\varepsilon, \tilde{u}) & \mapsto & \tilde{y}(T_0) - \varepsilon y^1 \end{array}$$

where $\tilde{y} : [0, T_0] \to \mathbb{R}^n$ is the solution of

$$\dot{\tilde{y}} = F(\tilde{y}) + \varepsilon A \tilde{y} + B\tilde{u}(t),\ \tilde{y}(0) = \varepsilon y^0. \tag{1}$$

The map G is of class C^1 in a neighborhood of $(0, \bar{u})$. One has $G(0, \bar{u}) = 0$. Moreover $G'_{\tilde{u}}(0, \bar{u})v = y(T_0)$ where $y : [0, T_0] \to \mathbb{R}^n$ is the solution of $\dot{y} = F'(\bar{y})y + Bv$, $y(0) = 0$. Hence $G'_{\tilde{u}}(0, \bar{u})$ is onto. Therefore there exist $\varepsilon_0 > 0$ and a C^1-map $\varepsilon \in (-\varepsilon_0, \varepsilon_0) \mapsto \tilde{u}^\varepsilon \in L^\infty((0, T_0); \mathbb{R}^m)$ such that

$$G(\varepsilon, \tilde{u}^\varepsilon) = 0,\ \forall \varepsilon \in (-\varepsilon_0, \varepsilon_0),$$
$$\tilde{u}^0 = \bar{u}.$$

Let $\tilde{y}^\varepsilon : [0, T_0] \to \mathbb{R}^n$ be the solution of the Cauchy problem $\dot{\tilde{y}}^\varepsilon = F(\tilde{y}^\varepsilon) + \varepsilon A \tilde{y}^\varepsilon + B\tilde{u}^\varepsilon(t)$, $\tilde{y}^\varepsilon(0) = \varepsilon y^0$. Then $\tilde{y}^\varepsilon(T_0) = \varepsilon y^1$. Let $y : [0, \varepsilon T_0] \to \mathbb{R}^n$ and $u : [0, \varepsilon T_0] \to \mathbb{R}^m$ be defined by

$$y(t) := \frac{1}{\varepsilon}\tilde{y}^\varepsilon\left(\frac{t}{\varepsilon}\right),\ u(t) := \frac{1}{\varepsilon^2}\tilde{y}^\varepsilon\left(\frac{t}{\varepsilon}\right).$$

Then $\dot{y} = F(y) + Ay + Bu$, $y(0) = y^0$ and $y(\varepsilon T_0) = y^1$. This concludes the proof of the controllability theorem if T is small enough. If T is not small, it suffices, with $\varepsilon > 0$ small enough, to go from y^0 to 0 during the interval of time $[0, \varepsilon T_0]$, stay at 0 during the interval of time $[\varepsilon T_0, T - \varepsilon T_0]$ and finally go from 0 to y^1 during the interval of time $[T - \varepsilon, T]$ (reverse the time).

A drawback of this strategy

However this strategy has a serious drawback in the case of partial differential equations if "Ay requires more derivatives on y that $F(y)$". For example it seems difficult to deduce from the controllability of

$$y_t + y_x = 0,\ y(t,0) = u(t),\ x \in (0,L), \tag{1}$$

in time $T > L$ the (null) controllability of

$$y_t + y_x - \varepsilon y_{xx} = 0,\ y(t,0) = u(t),\ y(t,L) = v(t),\ x \in (0,L), \tag{2}$$

in time $T > L$ if $\varepsilon > 0$ is small enough. So, let us propose a slightly different strategy (requiring stronger assumptions).

A slightly different strategy

Let us, moreover, assume that the control system

$$\dot{y} = Ay + F(y) + Bu \tag{1}$$

where the state is $y \in \mathbb{R}^n$ and the control is $u \in \mathbb{R}^m$ is locally controllable in small time. Then one can proceed in the following way in order to get the global null controllability in small time of $\dot{y} = Ay + F(y) + Bu$. We want to send y^0 to 0 to 0 in small time by using a suitable control u. Again we perform the following scaling

$$z(t) := \varepsilon y(\varepsilon t),\ w(t) := \varepsilon^2 u(\varepsilon t). \tag{2}$$

Then $\dot{y} = Ay + F(y) + Bu$ is equivalent to $\dot{z} = \varepsilon Az + F(z) + Bw$. We then look for z and v of the following form

$$z = \bar{y} + \varepsilon z^1 + \varepsilon^2 z^2 + \dots,\ w = \bar{u} + \varepsilon v^1 + \varepsilon^2 v^2 + \dots \tag{3}$$

Then, identifying the orders in ε^p, $p \in \{0,1\}$ in $\dot{z} = \varepsilon z + F(z) + Bw$ one gets

$$\dot{\bar{y}} = F(\bar{y}) + B\bar{u}, \tag{1}$$

$$\dot{z}^1 = A\bar{y} + \frac{\partial F}{\partial y}(\bar{y},\bar{u})z^1 + \frac{\partial F}{\partial u}(\bar{y},\bar{u})w^1. \tag{2}$$

Note that, from our assumption on $(\bar{y},\bar{u})$, (1) holds. For the initial data, we have

$$\bar{z}(0) = 0,\ z^1(0) = y^0. \tag{3}$$

From (1) and the properties of $(\bar{y},\bar{u})$, one has $\bar{y}(T_0) = 0$. From our assumption of controllability of the linearized control around $(\bar{y},\bar{u})$ one gets the existence of v^1 such that $z^1(T_0) = 0$. So, with this w^1, $z(T_0)$ is of order ε^2. Going back to the y variable one gets that $y(\varepsilon T_0)$ is of order ε. Then using the local controllability in small time of $\dot{y} = Ay + F(y) + Bu$, one gets that, for every $\tau > 0$, we can find a control allowing us to go for the control system $\dot{y} = Ay + F(y) + Bu$ from $y(\varepsilon T_0)$ to 0 during the interval of time $[\varepsilon T_0, \varepsilon T_0 + \tau]$.

This gives again the global null controllability in small time of $\dot{y} = Ay + F(y) + Bu$. It requires an extra property, namely, the local null controllability in small time of $\dot{y} = Ay + F(y) + Bu$, but it avoids the use of the inverse mapping theorem which is a serious problem in the pde framework if "Ay requires more derivatives on y that $F(y)$".

Morality

The "morality" behind these strategies is that the quadratic term $F(y)$ is the leading term compared to the linear term Ay for the global controllability: Ay is just an annoying perturbations (which can however be used when we are close enough to 0).
Of course, as one can see by looking at the proof of the controllability theorem, this method works only if we have a (good) convergence of the solution of the Navier-Stokes equations to the solution of the Euler equations when the viscosity tends to 0. This is the case on manifolds without boundary, which, in our situation, corresponds to the case where the control is on the full boundary of Ω: $\Gamma := \partial\Omega$ (or in the case of interior control on a manifold without boundary).

Let us recall that this convergence is not known (and might be wrong...) even in dimension $d = 2$ if there is no control. More precisely, let us assume that Ω is of class C^∞, that $d = 2$ and that $y^0 \in C_0^\infty(\Omega; \mathbb{R}^2)$ is such that div $y^0 = 0$. Let $T > 0$. Let $y \in C^\infty([0,T] \times \overline{\Omega}; \mathbb{R}^2)$ and $p \in C^\infty([0,T] \times \overline{\Omega})$ be the solution to the Euler equations

$$(E) \begin{cases} y_t + (y \cdot \nabla) y + \nabla p = 0, \text{ div } y = 0, \text{ in } (0,T) \times \Omega, \\ y \cdot n = 0 \text{ on } [0,T] \times \partial\Omega, \\ y(0, \cdot) = y^0 \text{ on } \overline{\Omega}. \end{cases}$$

Let $\varepsilon \in (0,1]$. Let $y^\varepsilon \in C^\infty([0,T] \times \overline{\Omega}; \mathbb{R}^2)$ and $p^\varepsilon \in C^\infty([0,T] \times \overline{\Omega})$ be the solution to the Navier-Stokes equations

$$(NS) \begin{cases} y_t^\varepsilon - \varepsilon \Delta y^\varepsilon + (y^\varepsilon \cdot \nabla) y^\varepsilon + \nabla p^\varepsilon = 0, \text{ div } y^\varepsilon = 0, \text{ in } (0,T) \times \Omega, \\ y^\varepsilon = 0 \text{ on } [0,T] \times \partial\Omega, \\ y(0, \cdot) = y^0 \text{ on } \overline{\Omega}. \end{cases}$$

One knows that there exists $C > 0$ such that $|y^\varepsilon|_{C^0([0,T]; L^2(\Omega;\mathbb{R}^2))} \leqslant C$, for every $\varepsilon \in (0,1]$.

One has the following challenging open problems.

Open problem (Convergence of Navier-Stokes to Euler)

(i) Does y^ε converge weakly to y in $L^2((0,T)\times\Omega;\mathbb{R}^2)$ as $\varepsilon\to 0^+$?

(ii) Let K be a compact subset of Ω and m be a positive integer. Does $y^\varepsilon_{|[0,T]\times K}$ converge to $y_{|[0,T]\times K}$ in $C^m([0,T]\times K;\mathbb{R}^2)$ as $\varepsilon\to 0^+$? (Of course, due to the difference of boundary conditions between the Euler equations and the Navier-Stokes equations, one does not have a positive answer to this last question if $K=\overline{\Omega}$.)

However this open problem is known to have a positive answer in the case of the Navier slip boundary condition. D. Iftimie and F. Sueur got in 2011 a rigorous boundary layer expansion in the case of the Navier slip boundary condition. This expansion is easier to handle than the Prandtl model (which deals with the Dirichlet boundary condition) because the main equation for the boundary layer correction is linear and well-posed. So there is some hope to be able to treat the case of the Navier slip boundary condition.

An advice due to L. Nirenberg

L. Nirenberg, besides to be a great mathematician, always give great advices when you have no more idea to solve a given problem. I was told that one of his famous advices is

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Have you tried the dimension 2?

Note that we are in dimension $d+1$. It means that we should take $d=1$ to follow Nirenberg's advice. At a first glance the 1-D analogous of the Navier-Stokes with full control on the boundary is the following Burgers control system

$$\begin{cases} y_t - y_{xx} + y y_x = 0, \\ y(t,0) = v(t),\ y(t,L) = w(t). \end{cases} \tag{1}$$

Unfortunately O. Imanuvilov and S. Guerrero proved in 2007 that this control system is not globally null controllable system in small time.

However the quadratic part (i.e. the "F part") of

$$y_t - y_{xx} + yy_x = 0,\ y(t,0) = v(t),\ y(t,L) = w(t) \tag{1}$$

is the following control system (in an implicit formulation: no boundary condition)

$$y_t + yy_x = 0, \tag{2}$$

which is not null controllable (even in large time and even locally). So (1) is not a good 1-D analogue of our Navier-Stokes control system since the Euler equations are controllable (JMC (1996), O. Glass (2002)).

A 1-D analogue of our Navier-Stokes control systems

In order to have a good 1-D analogue we add one more control and consider the following control system

$$(3) \qquad y_t - y_{xx} + yy_x = u(t),\ y(t,0) = v(t),\ y(t,L) = w(t),$$

(roughly speaking u plays the role of the pressure). Then, using the return method, M. Chapouly proved in 2009 that the quadratic part of (3), i.e. (in an implicit formulation)

$$(4) \qquad y_t + yy_x = u(t),$$

is globally null controllable in small time and then, from this result and the above "morality" she deduced the globally null controllability in small time of (3).

A case with a boundary layer

In M. Chapouly's case there was no boundary layer problem since one can do what we want on the boundary. The remaining challenging open problem was to remove the control w. The control system is then

$$y_t - y_{xx} + yy_x = u(t),\ y(t,0) = v(t),\ y(t,L) = 0. \tag{1}$$

and if one follows the above strategy there is a boundary layer which appears at $x = L$. This problem was solved by F. Marbach in 2014.

F. Marbach's method

The Burgers control system studied by F. Marbach is

$$(2) \qquad y_t - y_{xx} + yy_x = u(t),\ y(t,0) = v(t),\ y(t,L) = 0.$$

In 2014, F. Marbach proved the global null controllability in small time of the control system (2).

Theorem (F. Marbach (2014))

For every $T > 0$ and for every $y^0 \in L^2(0,L)$, there exist $u \in C^\infty([0,T])$ and $v \in C^\infty([0,T])$ such that the solution of

$$(3) \qquad \begin{cases} y_t - y_{xx} + \left(\dfrac{y^2}{2}\right)_x = u(t),\ t \in (0,T),\ x \in (0,L), \\ y(t,0) = v(t),\ y(t,L) = 0,\ t \in (0,T), \\ y(0,x) = y^0(x),\ x \in (0,L), \end{cases}$$

satisfies $y(T,\cdot) = 0$.

F. Marbach's proof

Note that the analogous of the Euler equation for (2) is (C. Bardos, A-Y. Le Roux and J.-C. Nédélec (1979))

$$\text{(4)} \qquad \begin{cases} y_t + (y^2/2)_x = u(t), \\ y(t,0) \in I(v(t)), \; y(t,1) \geq 0, \end{cases}$$

where $I(a) = (-\infty, 0]$ if $a \leqslant 0$ and $I(a) = (-\infty, -a) \cup \{a\}$ if $a > 0$. Using the return method as for the Euler equation, F. Marbach first proved that this control system is globally null controllable in small time. If ones uses the same control for the viscous Burgers equation a boundary layer appears at $x = L$. A fundamental result due to F. Marbach is that this boundary layer has a form which leads to a natural rapid dissipation. Then one can conclude by using a standard local null controllability result.

Key ingredients for our global controllability result for the Navier-Stokes equations with the Navier slip boundary condition

There are five main ingredients

1. The return method together with the idea to consider by scaling the Navier-Stokes as some kind of perturbation of the Euler equation (JMC (1992, 1996)),
2. The controllability of the Euler equation (JMC (1996), 0. Glass (2002)),
3. The description of the evolution of the boundary layer due to D. Iftimie and F. Sueur (2011),
4. The dissipation method due to F. Marbach (2014),
5. The local null controllability result due to S. Guerrero (2006).

Let us now give more details.

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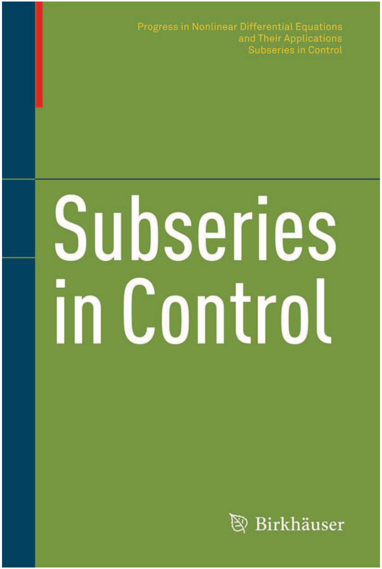

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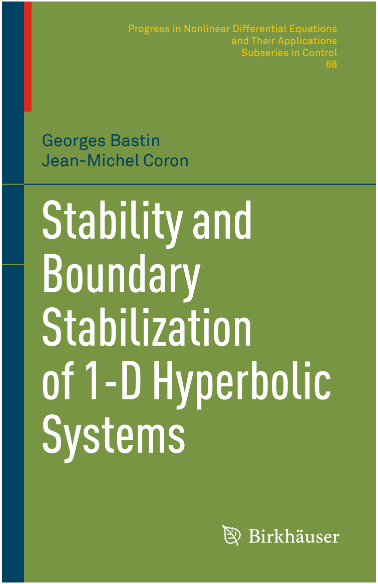

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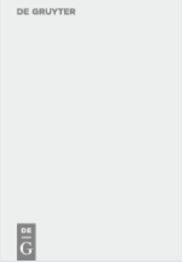


Going back to the details

As in finite dimension we perform the scaling: $z(t,x) := \varepsilon y(\varepsilon t, x)$ and $q(t,x) := \varepsilon^2 p(\varepsilon t, x)$. Now, (z,q) is the solution to the following system for $t \in (0,T)$:

$$
(\text{N-S}_\varepsilon) \qquad \begin{cases} z_t + (z\cdot\nabla)\,z - \varepsilon\Delta z + \nabla q = 0 & \text{on } (0,T)\times\Omega, \\ \operatorname{div} z = 0 & \text{on } (0,T)\times\Omega, \\ z\cdot n = 0 & \text{on } (0,T)\times(\partial\Omega\setminus\Gamma), \\ z\cdot n = 0,\ [D(z)n + Az]_{\tan} = 0 & \text{on } (0,T)\times(\partial\Omega\setminus\Gamma), \\ z|_{t=0} = \varepsilon y^0 & \text{on } \Omega. \end{cases}
$$

Due to the scaling chosen, we need to prove that we can obtain $|z(T,\cdot)|_{L^2(\Omega)} = o(\varepsilon)$ if we want to achieve global approximate null controllability (and then conclude by using the local null controllability). Since ε is small, we expect z to converge to the solution of the Euler equation. Hence, as in finite dimension, we first introduce the following asymptotic expansion for z:

$$z(t,x) = \bar{y}(t,x) + \varepsilon z^1(t,x) + \ldots. \tag{1}$$

The pressure is also expanded as:

$$q(t,x) = \bar{p}(t,x) + \varepsilon q^1(t,x) + \ldots. \tag{2}$$

The Euler equation

At order $\mathcal{O}(1)$, the first part $(\bar{y}, \bar{p})$ of our expansion is a solution to the Euler equation. Hence, the pair $(\bar{y}, \bar{p})$ is a return-method trajectory of the Euler equation on $[0, T]$:

$$
(3) \quad \begin{cases} \bar{y}_t + (\bar{y} \cdot \nabla)\, \bar{y} + \nabla \bar{p} = 0, \text{ on } (0, T) \times \Omega, \\ \operatorname{div} \bar{y} = 0 \text{ on } (0, T) \times \Omega, \\ \bar{y} \cdot n = 0 \text{ on } (0, T) \times \partial\Omega \setminus \Gamma, \\ \bar{y}(0, \cdot) = \bar{y}(T, \cdot) = 0 \text{ in } \Omega, \\ \text{the linearized Euler control system around } (\bar{y}, \bar{p}) \text{ is controllable.} \end{cases}
$$

Such $(\bar{y}, \bar{p})$ are constructed by JMC (1996) in dimension 2 and by O. Glass (2000) in dimension 3. It is here that we use the assumption that Γ meets every connected component of $\partial\Omega$.

Construction of $(\bar{y}, \bar{p})$ for $d = 2$

Take $\theta : \overline{\Omega} \to \mathbb{R}$ such that

$$\Delta\theta = 0 \text{ in } \Omega,\ \frac{\partial\theta}{\partial n} = 0 \text{ on } \partial\Omega \setminus \Gamma.$$

Take $\alpha : [0, T] \to \mathbb{R}$ such that $\alpha(0) = \alpha(T) = 0$. Finally, define $(\bar{y}, \bar{p}) : [0, T] \times \overline{\Omega} \to \mathbb{R}^2 \times \mathbb{R}$ by

$$(1) \qquad \bar{y}(t, x) := \alpha(t)\nabla\theta(x),\ \bar{p}(t, x) := -\dot{\alpha}(t)\theta(x) - \frac{\alpha(t)^2}{2}|\nabla\theta(x)|^2.$$

Then $(\bar{y}, \bar{p})$ is a trajectory of the Euler control system which goes from 0 to 0.

Controllability of the linearized control system around $(\bar{y}, \bar{p})$ if $d = 2$

The linearized control system around $(\bar{y}, \bar{p})$ is

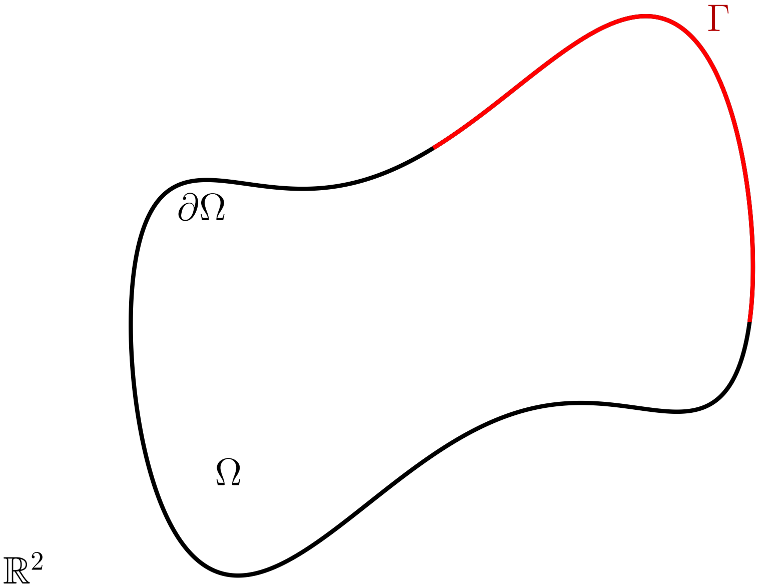
$$(1) \begin{cases} y_t + (\bar{y} \cdot \nabla)y + (y \cdot \nabla)\bar{y} + \nabla p = 0, \ \text{div } y = 0 \text{ in } [0,T] \times \Omega, \\ y \cdot n = 0 \text{ on } [0,T] \times (\partial\Omega \setminus \Gamma). \end{cases}$$

Again we assume that $d = 2$. Taking the curl of the first equation, one gets

$$(2) \qquad (\text{curl } y)_t + (\bar{y} \cdot \nabla)(\text{curl } y) = 0.$$

This is a simple transport equation on curl y. If there exists $a \in \overline{\Omega}$ such that $\nabla\theta(a) = 0$, then $\bar{y}(t,a) = 0$ and $(\text{curl } y)_t(t,a) = 0$ showing that (2) is not controllable. This is the only obstruction: If $\nabla\theta$ does not vanish in $\overline{\Omega}$, one can prove that (2) (and then (1)) is controllable if $\int_0^T \alpha(t)dt$ is large enough.

A good θ for $d = 2$ and Ω simply connected



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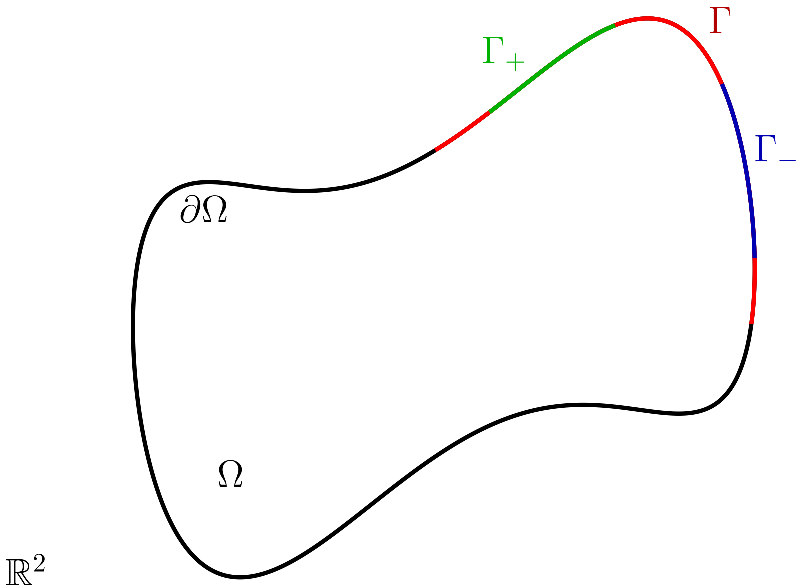

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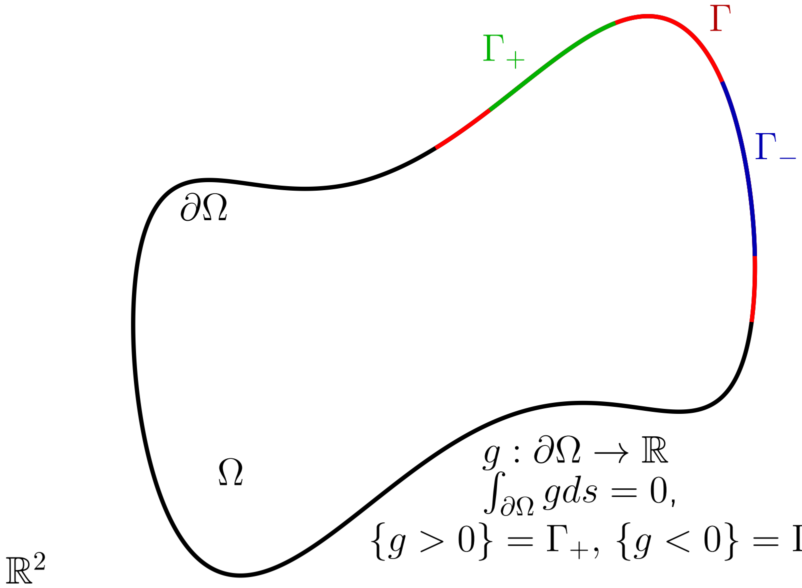
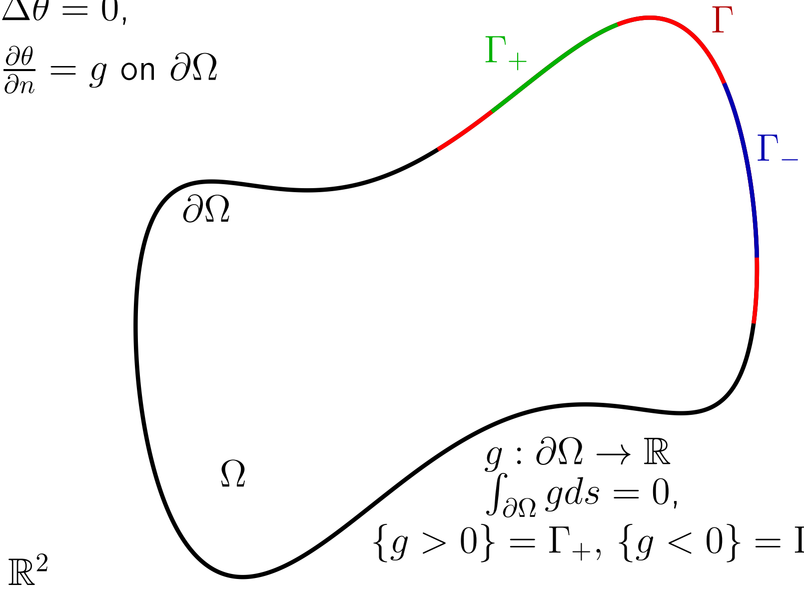


$g : \partial\Omega \to \mathbb{R}$
$\int_{\partial\Omega} g ds = 0,$
$\{g > 0\} = \Gamma_+, \{g < 0\} = \Gamma_-$

A good θ for $d = 2$ and Ω simply connected

$\Delta\theta = 0,$

$\frac{\partial\theta}{\partial n} = g$ on $\partial\Omega$



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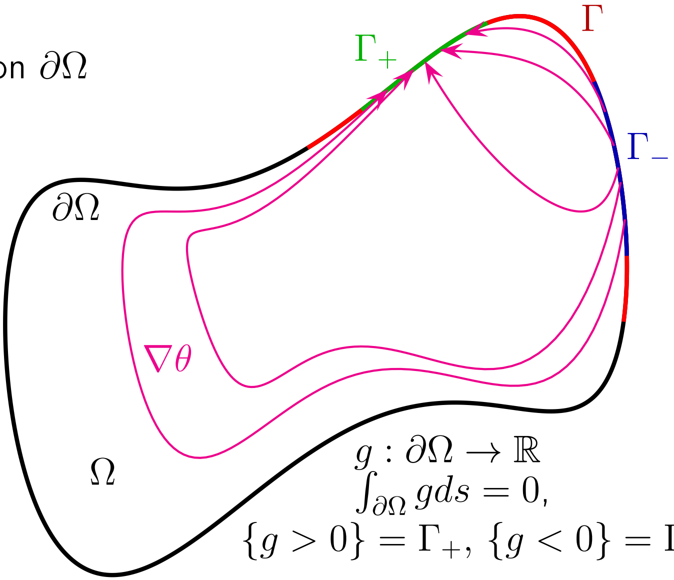

$g : \partial\Omega \to \mathbb{R}$

$\int_{\partial\Omega} g ds = 0,$

$\{g > 0\} = \Gamma_+,\ \{g < 0\} = \Gamma_-$

$\mathbb{R}^2$

Order 1

We move on to order $\mathcal{O}(\varepsilon)$. Here, the initial data y^0 comes into play. The equation of the order 1 is (using the fact that $\Delta \bar{y} = 0$)

$$
(1) \quad \begin{cases} z^1_t + (\bar{y} \cdot \nabla)\, z^1 + \left(z^1 \cdot \nabla\right) \bar{y} + \nabla q^1 = 0 & \text{in } \Omega \text{ for } t \geq 0, \\ \operatorname{div} z^1 = 0 & \text{in } \Omega \text{ for } t \geq 0, \\ z^1 \cdot n = 0 & \text{in } \partial\Omega \setminus \Gamma \text{ for } t \geq 0, \\ z^1(0, \cdot) = y^0 & \text{in } \Omega \text{ at } t = 0. \end{cases}
$$

This control system is controllable. So we can get $z^1(T_0) = 0$.

Boundary layer

Due to the scaling chosen, we need to prove that we can obtain $|z(T/\varepsilon,\cdot)|_{L^2(\Omega)} = o(\varepsilon)$. Following closely the original boundary layer expansion for Navier slip boundary conditions proved by D. Iftimie and F. Sueur (2011), the correct expansion for z:

$$z(t,x) = \bar{y}(t,x) + \sqrt{\varepsilon} v\left(t, x, \frac{\varphi(x)}{\sqrt{\varepsilon}}\right) + \varepsilon z^1(t,x) + \dots, \tag{2}$$

where $\varphi(x) := \mathrm{dist}(x, \partial\Omega)$ in a neighborhood of $\partial\Omega$. The pressure is again expanded as:

$$q(t,x) = \bar{p}(t,x) + \varepsilon q^1(t,x) + \dots. \tag{3}$$

Compared with the expansion we gave in finite dimension for $\dot{y} = Ay + F(y) + Bu$, expansion (2) introduces a boundary correction v. Indeed, $\bar{y}$ does not satisfy the Navier slip boundary condition on $\partial\Omega \setminus \Gamma$. The purpose of the second term v is to recover this boundary condition by introducing the tangential boundary layer generated by $\bar{y}$. More precisely, since the Euler system is a first-order system, we have only been able to impose one scalar boundary condition, namely, $\bar{y} \cdot n = 0$ on $\partial\Omega \setminus \Gamma$. Hence, the full Navier slip boundary condition is not satisfied by $\bar{y}$. Therefore, at order $\mathcal{O}(\sqrt{\varepsilon})$, we introduce a boundary layer correction v. This correction if fully tangential and has no normal part. This profile is expressed in terms both of the slow space variable $x \in \Omega$ and a fast scalar variable $\xi = \varphi(x)/\sqrt{\varepsilon}$. For $x \in \Omega$, $\varphi(x) \geq 0$. Thus, ξ lives in $\mathbb{R}_+$. As in D. Iftimie and F. Sueur (2011), v is the solution to:

$$
\text{(4)} \quad \begin{cases} v_t + [(\bar{y} \cdot \nabla)v + (v \cdot \nabla)\bar{y}]_{\tan} + \kappa\xi v_\xi - v_{\xi\xi} = 0 & \text{in } \mathbb{R}_+ \times \Omega,\ t \geq 0, \\ v_z(t, x, 0) = g^0(t, x) & \text{in } \{0\} \times \Omega,\ t \geq 0, \\ v(0, \cdot, \cdot) = 0 & \text{in } \mathbb{R}_+ \times \Omega,\ t = 0, \end{cases}
$$

where

$$
\text{(5)} \quad \kappa(t, x) := \frac{\bar{y}(t, x) \cdot n(x)}{\varphi(x)}, \qquad \text{in } [0, T_0] \times \Omega,
$$

$$
\text{(6)} \quad g^0(t, x) := 2\chi(x) \left[D\left(\bar{y}(t, x)\right) n(x) + A\bar{y}(t, x)\right]_{\tan} \quad \text{in } [0, T_0] \times \Omega.
$$

After time T_0, the boundary layer equation reduces to the following heat equations on the half line $\xi \geq 0$ (where the slow variable x plays the role of a parameter):

$$\text{(7)} \qquad \begin{cases} v_t - v_{\xi\xi} = 0, & \text{in } \mathbb{R}_+ \times \Omega \quad \text{for } t \geq T_0, \\ v_\xi(t,x,0) = 0 & \text{in } \{0\} \times \Omega \quad \text{for } t \geq T_0. \end{cases}$$

There is no more control. There is a natural dissipation on $[T_0, T/\varepsilon]$. However this dissipation is not good enough to get that the boundary profile v at the final time is small enough to apply a local controllability result and that the source terms generated by v in the equation satisfied by the remainder are integrable with respect to time. However this dissipation on $[T_0, T/\varepsilon]$ turns out to be good enough if if the function v at time T_0 satisfies the following moment properties

$$\text{(8)} \qquad \int_0^{+\infty} \xi^k v(T_0, x, \xi) d\xi = 0, \, \forall x \in \Omega, \, \forall k \in \{0,1,2,3\}.$$

Property (8) can be obtained by using controllability properties of the boundary layer equation during the interval of time $[0, T_0]$ (even if this controllability is not sufficient to get $v(T_0, \cdot, \cdot) = 0$ since $\xi \in [0, +\infty)$). This concludes the proof.

An open problem on Γ

Does the global controllability in small time of our Navier-Stokes control system holds if one replaces the assumption "Γ is an open subset of $\partial\Omega$ which meets every connected component of $\partial\Omega$" by the weaker assumption

$$(1) \qquad \Gamma \text{ is a non empty open subset of } \partial\Omega?$$

Note that (1) is sufficient for the local controllability of Navier-Stokes control system (Guerrero (2006)). It is also sufficient to get a global approximate controllability result in small time for the Euler control system, with exact controllability inside Ω (JMC (1996), Glass (2000, 2001)). However there is a difficulty to get a well-prepared boundary layer (i.e. a boundary layer which dissipates fast enough). A related question is what about the case of interior control?

An open problem on the regularity of the solution

Is it possible to get the global null controllability in the framework of strong solutions instead of weak solutions if the initial data is smooth? This problem is open for $d = 3$ only. Note that in the interval of time $[0, \varepsilon T_0]$ the solution is strong (if the initial data is smooth). It is during the dissipation stage that we do not know if the solution remains a strong solution.

Global controllability in small time of Korteweg de Vries equations

Let us start with the following

$$
(1) \qquad \begin{cases} y_t + y_x + y_{xxx} + yy_x = a(t),\ x \in (0, L) \\ y(t,0) = b(t),\ y(t,L) = c(t),\ y(t,L) = d(t), \end{cases}
$$

where, at time $t \in [0, T]$, the state is $y(t, \cdot) \in L^2(0, L)$ and the control is $(a(t), b(t), c(t), d(t))^{\mathrm{tr}} \in \mathbb{R}^4$. Using the approach of the global controllability in small time of the Navier-Stokes control system, M. Chapouly proved in 2009 that (1) is globally null controllable in small time. What happen to this global controllability in small time in the following situations.

1. One removes the control on the left: $b(t) = 0$. Then if we follow the return method strategy a boundary layer appears at $x = 0$. Is it possible to handle it (using F. Marbach's strategy)?
2. One removes one or two controls on the right: $c(t) = 0$ or/and $d(t) = 0$. Then, if we follow the return method strategy a boundary layer appears at $x = L$. Is it possible to handle it?
3. One removes the control $a(t)$. Note that in this the quadratic part (the leading term at infinity)

$$y_t + yy_x = 0,\ x \in (0, L) \tag{1}$$

which is not null controllable. Let us recall that for the analogous viscous Burgers equation, O. Imanuvilov and S. Guerrero proved in 2007 that this global controllability do not hold. However their proof relies heavily on the maximum principle, a maximum principle which do not hold for our KdV equation. Does the global controllability in small time holds? The corresponding Lagrangian controllability holds (L. Gagnon 2016).