Invariance for semilinear stochastic PDEs

Giuseppe Da Prato
Scuola Normale Superiore, Pisa (Italy)

Joint work with
Piermarco Cannarsa and Hélène Frankowska
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Plan of the talk

The first part of the talk is devoted to deterministic evolution equations in a Hilbert spaces H of the form

$$\begin{cases} X'(t) = AX(t) + B(X(t)), \\ X(0) = x \in H, \end{cases} \qquad (PDE)$$

where $A : D(A) \subset H$ is linear and $F : H \to H$ is nonlinear.

In the second part, to take into account random perturbations, we will add to equation (PDE) a stochastic term of the form $\sigma(X(t))dW(t)$, where W is an H–valued cylindrical Wiener process.

In both cases we shall present necessary and sufficient conditions for the invariance of a closed convex set K.

The deterministic case

Hypothesis 1

(i) $A : D(A) \subset H \to H$ *is the infinitesimal generator of a strongly continuous semigroup* e^{tA} *and there is* $\omega \in \mathbb{R}$ *such that*

$$\langle Ax, x \rangle \leq \omega |x|^2, \quad \forall\, x \in D(A). \tag{1}$$

(ii) $B : H \to H$ *is continuous and quasi–dissipative, i.e. there exists* $M \in \mathbb{R}$ *such that*

$$\langle B(x) - B(y), x - y \rangle \leq M|x - y|^2, \quad \forall\, x, y \in H. \tag{2}$$

Under Hypothesis 1 it is well known that for any $x \in H$ there exists a unique continuous solution to the integral equation

$$X(t,x) = e^{tA}x + \int_0^t e^{(t-s)A}B(X(s,x))ds, \quad t \geq 0,$$

called a mild solution to problem (PDE).

Let now $K \subset H$ be a non empty, closed and convex subset of H (possibly with a non empty interior); we aim to find necessary and sufficient conditions in order that

$$x \in K \Rightarrow X(t,x) \in K, \quad \forall\, t \geq 0.$$

In this case we say that K is invariant for the dynamical system (PDE).

Our analysis will be based on the distance function:

$$d_K(x) = \min_{y \in K} |x - y|, \quad \forall\, x \in H.$$

Since K is convex, the minimum above is attained at a unique point $\Pi_K(x)$, the projection of x over K, so that

$$d_K(x) = |x - \Pi_K(x)|, \quad \forall\, x \in H.$$

We recall that d_K is Lipschitz:

$$|d_K(x) - d_K(y)| \leq |x - y|, \quad \forall\, x, y \in H$$

and that the function:

$$V_K(x) := d_K^2(x), \quad x \in H,$$

is of class $C^{1,Lip}$ and it results

$$DV_K(x) = 2(x - \Pi_K(x)), \quad \forall\, x \in H.$$

Theorem 1 ([CaDaFr16])

Under Hypothesis 1 the following assertions are equivalent:

(i) K *is invariant.*

(ii) There is $N \geq 0$ *such that*

$$\langle DV_K(x), Ax + B(x)\rangle \leq NV_K(x), \quad \forall\, x \in D(A) \cap K^c. \tag{3}$$

Notice that condition (3) can be written as

$$2\langle x - \Pi_K(x), Ax + B(x)\rangle \leq NV_K(x), \quad \forall\, x \in D(A) \cap K^c. \tag{4}$$

Sketch of the proof

$(i) \Rightarrow (ii)$. Assume that K is invariant and let $x \in D(A)$. Then

$$\begin{aligned} &\frac{1}{t}\left(V_K(X(t,x)) - V_K(x)\right) \\ &= \frac{1}{t}\left(V_K(X(t,x)) - V_K(X(t,\Pi_K(x))) - V_K(x)\right), \end{aligned} \tag{5}$$

because $V_K(X(t,\Pi_K(x))) = 0$. Moreover, taking into account that d_K is Lipschitz, yields

$$\begin{aligned} &\frac{1}{t}\left(V_K(X(t,x)) - V_K(x)\right) \\ &\leq \frac{1}{t}\left[d_K(X(t,x))|X(t,x)) - X(t,\Pi_K(x))| - V_K(x))\right]. \end{aligned} \tag{6}$$

Also, by Hypothesis 1 it follows that the dependence of $X(t,x)$ from x is Lipschitzian,

$$|X(t,x) - X(t,y)| \leq e^{(\omega+M)t}|x-y|, \quad \forall\, x,y \in H.$$

Therefore

$$\frac{1}{t}\,[(V_K(X(t,x)) - V_K(x))$$

$$\leq \frac{1}{t}\,[d_K(X(t,x))e^{(\omega+M)t}|x - \Pi_K(x))| - V_K(x))])$$

$$= \frac{1}{t}\,[d_K(X(t,x))d_K(x)e^{(\omega+M)t} - V_K(x))]).$$

Letting $t \to 0$, yields

$$\langle DV_K(x), Ax + B(x)\rangle \leq (\omega + M)V_K(x),$$

and the implication $(i) \Rightarrow (ii)$ is proved.

$(ii) \Rightarrow (i)$. Take $x \in D(A) \cap K$ and write (formally),

$$\frac{d}{dt} V_K(X(t,x)) = \langle DV_K(X(t,x)), AX(t,x) + B(X(t,x)) \rangle$$

By the assumption it follows that

$$\frac{d}{dt} V_K(X(t,x)) \leq N V_K(X(t,x)),$$

which implies

$$V_K(X(t,x)) \leq e^{Nt} V_K(x).$$

Since $V_K(x) = 0$ we have $V_K(X(t,x)) = 0$ so that

$$X(t,x) \in K, \quad \forall\, t \geq 0.$$

We can also give a necessary and sufficient condition for the invariance of K involving only the boundary ∂K.

Proposition 2 ([CaDaFr16])

Assume, besides Hypothesis 1, that $\Pi_K(x) \in D(A)$ for all $x \in D(A)$. Then the following assertions are equivalent :

(i) K is invariant.

(ii) There is $N \geq 0$ such that

$$\langle p, Ax + B(x) \rangle \leq 0 \quad \forall\, x \in D(A) \cap \partial K,\ \forall\, p \in \mathscr{N}_K, \tag{7}$$

where $\mathscr{N}_K$ is the normal cone of K.

Proof

$(i) \Rightarrow (ii)$. Assume that K is invariant and let $x \in D(A) \cap \partial K$. Recall that by Theorem 1 we have

$$2\langle x - \Pi_K(x), Ax + B(x)\rangle \leq NV_K(x), \quad \forall\, x \in D(A).$$

Now, replacing x with $x_\lambda = x + \lambda p,\ \lambda > 0$, and taking into account that $\Pi_K(x_\lambda) = x, \quad d_K(x_\lambda) = \lambda|p|$, yields,

$$\langle x_\lambda - \Pi_K(x_\lambda), Ax_\lambda + B(x_\lambda)\rangle = \lambda\langle p, Ax_\lambda + B(x_\lambda)\rangle \leq \frac{1}{2} N\lambda^2|p|^2.$$

Dividing both sides by λ and letting $\lambda \downarrow 0$, yields

$$\langle p, Ax + B(x)\rangle \leq 0,$$

as claimed.

$(ii) \Rightarrow (i)$. Assume conversely that

$$\langle p, Ax + B(x)\rangle \leq 0, \quad \forall\, x \in D(A) \cap \partial K,\ \forall\, p \in \mathscr{N}_K.$$

Let $x \in D(A)$ and set $y = \Pi_K(x)$. Then write

$$\langle x - \Pi_K(x), Ax + B(x)\rangle = \langle x - y, Ax + B(x)\rangle$$

$$= \langle x - y, Ay + B(y)\rangle + \langle x - y, A(x - y) + B(x) - B(y)\rangle$$

$$\leq (\omega + M)|x - y|^2 = (\omega + M)|x - y|^2 V_K(x),$$

which yields

$$2\langle x - \Pi_K(x), Ax + B(x)\rangle \leq N V_K(x), \quad \forall\, x \in D(A) \cap K^c.$$

Therefore K is invariant in view of Theorem 1.

Example 1. The unitary ball

Let $K = B_1 = \{x \in H : \ |x| \leq 1\}$. Then we have

$$d_K(x) = (|x| - 1)\, \mathbb{1}_{\{|x|>1\}},$$

$$\Pi_K(x) = x \text{ if } x \in B_1, \quad \Pi_K(x) = \frac{x}{|x|} \text{ if } x \notin B_1.$$

Therefore, if $x \in D(A)$ we have $\Pi_K(x) \in D(A)$ and the normal cone $\mathscr{N}_K(x)$ at $x \in \partial B_1$ is given by

$$\mathscr{N}_K(x) = \{\lambda x : \ \lambda \geq 0\}.$$

By Proposition 2 it follows that B_1 is invariant if and only if

$$\langle x, Ax + B(x)\rangle \leq 0, \quad \forall\, x \in D(A),\ |x| = 1.$$

Example 2. The cone of nonnegative functions

Let $H = L^2(\mathscr{O})$ where $\mathscr{O}$ is an open subset of $\mathbb{R}^d$ with a regular boundary $\partial\mathscr{O}$ and

$$K = \{x \in L^2(\mathscr{O}) : \ x(\xi) \geq 0, \ \text{a.e.}\}.$$

K has an empty interior. Moreover,

$$V_K(x) = |x^-|^2 = \int_{\mathscr{O}} |x^-(\xi)|^2 \, d\xi, \quad x^- = \max\{0, -x\}.$$

So, $DV_K(x) = -2x^-$, and the iff condition for the invariance

$$\langle DV_K(x), Ax + B(x)\rangle \leq NV_K(x), \quad \forall\, x \in D(A) \cap K^c,$$

reduces to

$$-2\langle x^-, Ax + B(x)\rangle \leq N|x^-|^2, \quad \forall\, x \in D(A). \tag{8}$$

Let us now consider in particular the Heat equation, taking $A = \Delta$ (the Laplacian) equipped with Dirichlet boundary conditions.

Since for $x \in D(A) = H^2(\mathscr{O}) \cap H^1_0(\mathscr{O})$ we have

$$\langle x^-, Ax \rangle = -|\nabla x|^2, \quad \langle x^-, B(x) \rangle = \langle x^-, B(-x^-) \rangle,$$

condition

$$-2\langle x^-, Ax + B(x) \rangle \leq N|x^-|^2, \quad \forall\, x \in D(A).$$

is equivalent to

$$-2\langle x^-, B(x^-) \rangle \leq N|x^-|^2 + |\nabla x|^2, \quad \forall\, x \in D(A). \tag{9}$$

We note that if $B(0) = 0$ this condition is obviously fulfilled.

Remark

In the paper by [CaDaFr16], Theorem 1 and Proposition 2 above are proved in more general situations including non empty closed sets K which are not necessarily convex as well as by replacing the Hilbert space H by a Banach space X.

In this way reaction–diffusion equations with polynomial nonlinearity are covered by our results.

When H is infinite dimensional and A is unbounded, several sufficient conditions for the invariance are available in the literature, but necessary conditions are laking, except some one which seems not easy to check, see [CaNeVr07] .

We stress that necessary conditions are important in the applications, in particular to scientific modelling. ■

The stochastic case

As we said before, to take into account random perturbations, one is lead to add to equation (PDE) a stochastic term of the form

$$\sigma(X(t))dW(t),$$

where W is an H–valued cylindrical Wiener process in some filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$. Then problem (PDE) reduces to a stochastic PDE:

$$\begin{cases} dX(t) = (AX(t) + B(X(t)))dt + \sigma(X(t))dW(t), \\ X(0) = x \in H. \end{cases} \qquad (SPDE)$$

Hypothesis 2

We assume, besides Hypothesis 1, that

$$\sigma : H \to \mathscr{L}_2(H)$$

is Lipschitz continuous, where $\mathscr{L}_2(H)$ *is the Hilbert space of all Hilbert–Schmidt operators on* H*, endowed with the scalar product*

$$\langle T, S \rangle_{\mathscr{L}_2(H)} = \mathrm{Tr}\,[TS^*].$$

As well known, under Hypothesis 2, there exists a unique mild solution $X(\cdot, x)$ to problem (SPDE) for any $x \in H$.

$X(\cdot, x)$ is the unique mean-square continuous and adapted stochastic process solving the integral equation

$$\begin{aligned} X(t,x) &= e^{tA}x + \int_0^t e^{(t-s)A} B(X(s,x))ds \\ &\quad + \int_0^t e^{(t-s)A}\sigma(X(s,x))dW(s), \quad t \geq 0. \end{aligned}$$

We are also given a non empty, closed convex set K as before and we look for necessary and sufficient conditions such that

$$x \in K \Rightarrow X(t,x) \in K, \quad \forall\, t \geq 0, \ \mathbb{P}\text{–a.s.}.$$

In this case we say that K is invariant for (SPDE).

A problem arises, however, in applying Itô's formula to $d_K^2(X(t,x))$ because d_K^2 is in general only $C^{1,Lip}$, whereas Itô's formula requires a C^2 regularity.

For this reasons we replace the square with the forth power of the distance setting $W_K = d_K^4$.

It happens that d_K^4 is of class C^2 in some interesting situations as for instance when K is a ball, a closed subspace or a half–space.

If K is the set of all positive function, d_K^4 is not of class C^2, but it can be slightly changed following [CaDa12], in order that the theory below applies.

So, we shall assume, besides Hypothesis 2, that

Hypothesis 3

$W_K := d_K^4 \in C^2(H)$.

Notice that

$$DW_K(x) = 4d_K^2(x)(x - \Pi_K(x)),$$

and that, setting

$$n(x) := \frac{x - \Pi_K(x)}{d_K(x)} = Dd_K(x), \quad x \in K^c,$$

we have for $x \in K^c$,

$$D^2 W_K(x) = 12 d_K(x)^2 n(x) \otimes n(x) + 4 d_K^3(x) n'(x).$$

Applying Itô's formula to $W_K(X(t,x))$ we deduce that

$$\frac{d}{dt}\,\mathbb{E}[W_K(X(t,x))] = \mathbb{E}[\mathscr{L}W_K(X(t,x))],$$

where the Kolmogorov operator $\mathscr{L}$ is given by

$$\begin{aligned}\mathscr{L}W_K(x) &= 2d_K^3(x)\,\mathrm{Tr}\,[n'(x)a(x)] + 6d_K^2(x)\langle a(x)n(x), n(x)\rangle \\ &\quad +4d_K^3(x)\langle Ax + B(x), n(x)\rangle, \qquad x \in D(A)\cap K^c\end{aligned}$$

and $a(x) := \sigma(x)\sigma^*(x),\ x \in H$.

Theorem 2 ([CaDaFr])

Under Hypotheses 2 and 3 the following assertions are equivalent:

(i) K *is invariant.*

(ii) There is $N > 0$ *such that*

$$\mathscr{L} W_K(x) \leq N W_K(x), \quad \forall\, x \in D(A) \cap K^c. \tag{10}$$

Sketch of the proof

$(i) \Rightarrow (ii)$. Assume that K is invariant and let $x \in D(A)$. Write

$$\begin{aligned} &\frac{1}{t}\left(W_K(X(t,x)) - W_K(x)\right) \\ &= \tfrac{1}{t}\left(W_K(X(t,x)) - W_K(X(t,\Pi_K(x))) - W_K(x)\right) \end{aligned} \tag{11}$$

Using lipschitzianity of d_k, yields

$$\begin{aligned} &\frac{1}{t}\left(V_K(X(t,x)) - V_K(x)\right) \\ &\leq \frac{1}{t}\left[d_K^3(X(t,x))|X(t,x)) - X(t,\Pi_K(x))| - V_K(x)\right]. \end{aligned} \tag{12}$$

Since the dependence of $X(t,x)$ from x is Lpschitzian we have

$$\mathbb{E}(|X(t,x)-X(t,y)|^2) \leq e^{2(\omega+M)t}|x-y|^2, \quad x,y \in H \qquad (13)$$

and we deduce that

$$\frac{1}{t}\,\mathbb{E}[(W_K(X(t,x)) - V_K(x))] \leq \frac{1}{t}\,e^{(\omega+M)t} d_K^3(x)\,|e^{(\omega+M)t}-1|\,W_K(x).$$

Using Itô's formula and letting $t \to 0$, yields

$$\mathscr{L} W_K(x) \leq (\omega+M) W_K(x),$$

and the conclusion follows.

$(ii) \Rightarrow (i)$. Take $x \in D(A) \cap K$ and write,

$$\frac{d}{dt}\mathbb{E}[W_K(X(t,x))] = \mathbb{E}[\mathscr{L} W_K(X(t,x))].$$

Then by (10) it follows that

$$\frac{d}{dt}\mathbb{E}[V_K(X(t,x))] \leq N\mathbb{E}[V_K(X(t,x))],$$

so that

$$\mathbb{E}[V_K(X(t,x))] \leq e^{Nt} V_K(x) = 0,$$

which implies

$$X(t,x) \in K, \quad \mathbb{P}\text{–a.s.}, \ \forall\, t \geq 0.$$

We can also give a necessary and sufficient condition for the invariance of K involving only the boundary ∂K when the signed distance $\overline{d}_K$ defined by

$$\overline{d}_K(x) = d_K(x) - d_{\overline{K^c}}(x), \quad x \in H,$$

is of class $C^{2,Lip}$ (this requires K to be the closure of its interior).

Proposition 2 ([CaDaFr])

Assume that $\overline{d}_K$ is of class $C^{2,Lip}$ and that $\Pi_K \in D(A)$ for all $x \in D(A)$. Then K is invariant if and only if

$$\begin{cases} (i) \ \ \langle a(x)n(x), n(x)\rangle = 0, \quad \forall\, x \in D(A) \cap \partial K. \\ (ii)\, \mathrm{Tr}\,[n'(x)a(x)] + 2\langle Ax + B(x), n(x)\rangle \leq 0, \ \forall\, x \in D(A) \cap \partial K. \end{cases}$$

Example. The unitary ball

Let $K = B(0,1) = \{x \in H : \ |x| \leq 1\}$. *Then*

$$d_K(x) = (|x| - 1)\mathbb{1}_{|x|>1\}},$$

$$W_K(x) = (|x| - 1)^4 \mathbb{1}_{|x|>1\}},$$

$$n(x) = \frac{x}{|x|}, \quad n'(x) = \frac{1}{|x|} - \frac{x \otimes x}{|x|^3}, \quad |x| \geq 1.$$

Then by Proposition 2, K *is invariant if and only if*

$$\begin{cases} (i) \ \ \langle a(x)x, x\rangle = 0, \quad \forall\, x \in D(A) \cap \partial K. \\ (ii) \ \ \mathrm{Tr}\,[a(x)] + 2\langle Ax + B(x), x\rangle \leq 0, \ \forall\, x \in D(A) \cap \partial K. \end{cases}$$

Example. Invariance of a subspace

Let Z be a closed subspace of H and let P be the orthogonal projector onto Z see [Fi06].

Then, $\Pi_Z(x) = Px$,

$$n(x) = \frac{x - Px}{|x - Px|}, \quad x \notin Z$$

and

$$Dn(x) = \frac{I - P}{|x - Px|} - \frac{(x - Px) \otimes (x - Px)}{|x - Px|^3}, \quad x \notin Z,$$

By Theorem 2, Z is invariant if and only if

$$\frac{1}{2}|x - Px|^2 \operatorname{Tr}[a(Px)(I - P)] + |x - Px|^2 \langle b(Px)(x - Px)\rangle$$

$$+\langle a(Px)(x - Px), x - Px\rangle \leq 0, \quad \forall\, x \in H.$$

Example. Invariance of the set of nonnegative functions

Let $H = L^2(\mathscr{O})$ where $\mathscr{O}$ is an open subset of $\mathbb{R}^d$ with a regular boundary $\partial\mathscr{O}$. We set

$$K = \{x \in L^2(\mathscr{O}) : \ x(\xi) \geq 0, \ \text{a.e.}\}.$$

Then we have

$$d_K^2(x) = \int_{\mathscr{O}} [\min\{x, 0\}]^2 d\xi = \int_{\mathscr{O}} [x^-(\xi)]^2 d\xi \,. \tag{14}$$

Note that $d_K^4(x)$ fails to be C^2 on H, as it is easy to see. So, this example is not covered by Theorem 2.

Therefore, we shall replace, following [CaDa12], the fourth power of the distance by the function

$$V(x) := \int_{\mathscr{O}} F(x^-(\xi))d\xi, \quad x \in H, \tag{15}$$

where

$$F(r) = \begin{cases} r^4 & |r| \leq 1 \\ 6r^2 - 8|r| + 3 & |r| \geq 1. \end{cases}$$

Observe that F is convex on $\mathbb{R}$, and $F'(r) \geq 0$ for $r \geq 0$. Notice that V is of class C^1 but not of class C^2 because it only possesses weakly continuous Gateaux second derivatives.

But Itô's formula can be generalized ad we can characterize the invariance of K see [CaDa16].

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