Multiplicative controllability for semilinear reaction-diffusion equations

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(joint work with P. Cannarsa and A. Y. Khapalov)

INδAM WORKSHOP
"NEW TRENDS IN CONTROL THEORY AND PDEs "
ON THE OCCASION OF THE 60TH BIRTHDAY OF PIERMARCO CANNARSA
INδAM, 3 July 2017

I first saw the surname Cannarsa in 2007 when...

ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

S. CAMPANATO
P. CANNARSA
Differentiability and partial Hölder continuity of the solutions of non-linear elliptic systems of order $2m$ with quadratic growth

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4^e série, tome 8, nº 2 (1981), p. 285-309
<http://www.numdam.org/item?id=ASNSP_1981_4_8_2_285_0>

Campanato Cannarsa.pdf

Figure: S. Campanato - P. Cannarsa, Annali della Scuola Normale Superiore di Pisa, 1981

Outline

1. Introduction
   - Control theory
   - Additive vs multiplicative controllability
   - Obstruction to multiplicative controllability: nonnegative or sign changing states
     - State of art: Nonnegative controllability
2. Main results: multiplicative controllability for sign changing states
   - 1-D reaction-diffusion equations
     - Main ideas for the proof of the main result
   - m-D reaction-diffusion equations with radial symmetry
     - Problem formulation and main results
     - An idea of the proof
   - 1-D degenerate reaction-diffusion equations
3. Open problems

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Control theory & Reaction-diffusion equations

Reference

P. Cannarsa, G. F., A. Y. Khapalov, Multiplicative controllability for semilinear reaction-diffusion equations with finitely many changes of sign , To appear in Journal de Mathématiques Pures et Appliquées, ArXiv: 1510.04203.

$\Omega \subseteq \mathbb{R}^m$ bounded

$$\begin{cases} u_t = \Delta u + v(x,t)u + f(u) & \text{in} \quad Q_T := \Omega \times (0,T) \\ u_{|_{\partial\Omega}} = 0 & t \in (0,T) \\ u_{|_{t=0}} = u_0 \end{cases} \tag{1}$$

$v \in L^\infty(Q_T)$, $f : \mathbb{R} \to \mathbb{R}$ Lipschitz, $\exists f'(0)$ and $f(0) = 0$.

Strong maximum principle

$f(u)$ is differentiable at 0 and $f(0) = 0 \implies \frac{f(u)}{u} \in L^\infty(Q_T)$

Thus, the strong maximum principle (SMP) can be extend to semilinear parabolic equation:

$$u_t = \Delta u + \left(v + \frac{f(u)}{u}\right) u$$

Well-posedness result

$$u_0 \in L^2(\Omega) \Longrightarrow \exists! u \in L^2(0,T; H_0^1(\Omega)) \cap C([0,T]; L^2(\Omega));$$

$$u_0 \in H_0^1(\Omega) \implies u \in H^1(0,T; L^2(\Omega)) \cap C([0,T]; H_0^1(\Omega)) \cap L^2(0,T; H^2(\Omega)).$$

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Controllability: linear case ($f(u)=0$)

$$\begin{cases} u_t &= \Delta u + vu + h(x,t)\mathbb{1}_\omega \\ u_{|\partial\Omega} = 0 & (\omega\subset\Omega) \\ u_{|t=0} &= u_0 \end{cases}$$

Additive controls
(locally distributed source terms)

$$\begin{cases} u_t &= \Delta u + vu \\ u_{|\partial\Omega} = g(t) \\ u_{|t=0} &= u_0 \end{cases}$$

Boundary controls

$$\begin{cases} u_t &= \Delta u + v(x,t)u \\ u_{|\partial\Omega} = 0 \\ u_{|t=0} &= u_0 \end{cases}$$

Bilinear controls (global!)
(multiplicative controllability)

Definition (Exact controllability)

$\forall u_0\in H_0, u^*\in H^*, (H_0, H^*\subseteq L^2(\Omega)), \exists$ *"a control function"*, $T>0$ *such that* $u(\cdot,T)=u^*$.

Definition (Approximate controllability)

$\forall u_0\in H_0, u^*\in H^*, (H_0, H^*\subseteq L^2(\Omega)), \forall\varepsilon>0, \exists$ *"a control function"*, $T>0$ *such that*
$$\|u(\cdot,T)-u^*\|_{L^2(\Omega)}<\varepsilon.$$

Regularizing effect of the heat equation and obstruction to exact controllability:
$H^*\subset H_0=L^2(\Omega)$:

$$u_0\in L^2(\Omega)\Longrightarrow\exists! u\in L^2(0,T;H^1_0(\Omega))\cap C([0,T];L^2(\Omega));$$
$$\Longrightarrow u(\cdot,t)\in H^1_0(\Omega), \forall t>0.$$

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Outline

Additive vs multiplicative controllability

Additive controls

$$u_t \; - \; \Delta u \; = \; v(x,t)u \; + h(x,t)\mathbb{1}_{\omega}, \qquad \omega \subset \Omega$$

$\forall u_0 \in L^2(\Omega), u^* \in H($“suitable”$H \subset L^2(\Omega)), \exists \omega \subseteq \Omega, h, T > 0$ such that $u(\cdot, T) = u^*$.

Reference

H. Fattorini, D. Russell Exact controllability theorems for linear parabolic equations in one space dimension Arch. Rat. Mech. Anal., **4**, *(1971) 272–292*

Multiplicative controllability and Applied Mathematics

$\Rightarrow$ rather than $\downarrow$

$$u_t - \Delta u = \; v(x,t) \;\; u + h(x,t)$$

use $\uparrow$ as control variable

Remark

$$\Phi : \text{“control”} \longmapsto \text{“solution”}$$

Additive controls vs *Bilinear controls*

$\Phi : h \longmapsto u$ *is a linear map;* $\Phi : v \longmapsto u$ *is a nonlinear map.*

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- nonlinear problem: Nash-Moser theorem (Hörmander version);
- controllability of the linearized problem;
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Some references on bilinear control of PDEs

- Ball, Marsden and Slemrod (1982)
  [rod and wave equation]
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  [Scrödinger equation]
- Fernández (2001), Lin, Gao and Liu (2006)
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$$\begin{cases} u_t = \Delta u + v(x,t)u + f(u) & \text{in} \quad Q_T := \Omega \times (0,T) \\ u_{|\partial\Omega} = 0 & t \in (0,T) \\ u_{|t=0} = u_0 \end{cases}$$

Definition (Approximate controllability)

An evolution system is called globally approximately controllable, if any initial state u_0 in H_0 can be steered into any neighborhood of any target state $u^* \in H^*$ at time T, by a suitable control.

Strong Maximun Principle and obstruction to multiplicative controllability: $H^* \neq H_0^1(\Omega)$

$$u_0(x) = 0 \Longrightarrow u(x,t) = 0$$
$$u_0(x) \geq 0 \Longrightarrow u(x,t) \geq 0$$

If $u_0(x) \geq 0$ in Ω, then the SMP demands that the respective solution to (1) remains nonnegative at any moment of time, regardless of the choice of v. This means that system (1) cannot be steered from any nonnegative u_0 to any target state which is negative on a nonzero measure set in the space domain.

Controllability:

- Nonnegative states
- Sign changing states

$$\begin{cases} u_t \;=\; \Delta u \;+\; v(x,t)u \;+\; f(u) & \text{in} \quad Q_T := \Omega \times (0,T) \\ u_{|\partial\Omega} = 0 & t \in (0,T) \\ u_{|t=0} \;=\; u_0 & \end{cases}$$

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Definition

We say that the system (1) *is nonnegatively globally approximately controllable in* $L^2(\Omega)$, *if for every* $\eta > 0$ *and for any* $u_0,\ u^* \in L^2(\Omega),\ \ u_0,\ u^* \geq 0$, *with* $u_0 \neq 0$ *there are a* $T = T(\eta, u_0, u^*) \geq 0$ *and a bilinear control* $v \in L^\infty(Q_T)$ *such that for the corresponding solution* u *of* (1) *we obtain*

$$\|u(T,\cdot) - u^*\|_{L^2(\Omega)} \leq \eta\,.$$

Reference

P. Cannarsa, G. F., Approximate multiplicative controllability for degenerate parabolic problems with robin boundary conditions, , CAIM, (2011).

Reference

P. Cannarsa, G. F., Approximate controllability for linear degenerate parabolic problems with bilinear control, Proc. Evolution Equations and Materials with Memory 2010, vol. Sapienza Roma, 2011, pp. 19–36.

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Changes of sign

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We assume that $u_0 \in H^1_0(0,1)$ has *a finite number of points of sign change*, that is, there exist points $0 = x^0_0 < x^0_1 < \cdots < x^0_n < x^0_{n+1} = 1$ such that

$$u_0(x) = 0 \iff x = x^0_l, \quad l = 0,\dots,n+1.$$
$$u_0(x)u_0(y) < 0, \quad \forall x \in \left(x^0_{l-1}, x^0_l\right), \forall y \in \left(x^0_l, x^0_{l+1}\right).$$

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$$u_0(x) = 0 \quad \Longleftrightarrow \quad x = x_l^0, \quad l = 0,\ldots,n+1.$$

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Let $u_0 \in H_0^1(0,1)$ have *a finite number of points of sign change.*

Theorem (P. Cannarsa, G.F., A.Y. Khapalov, JMPA)

Consider any $u^* \in H_0^1(0,1)$ *which has exactly as many points of sign change in the same order as* u_0. *Then,*

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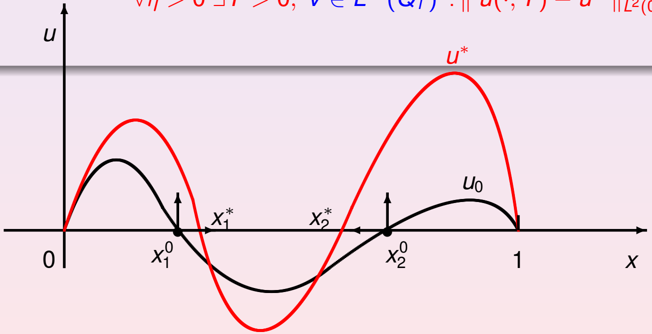

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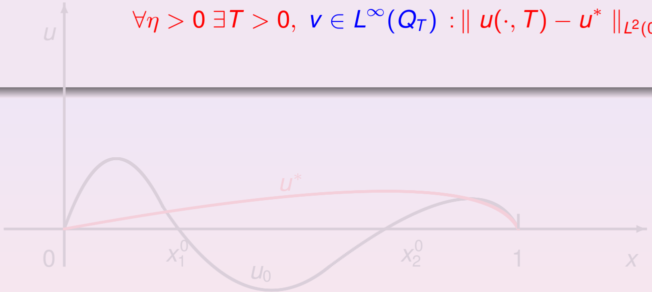

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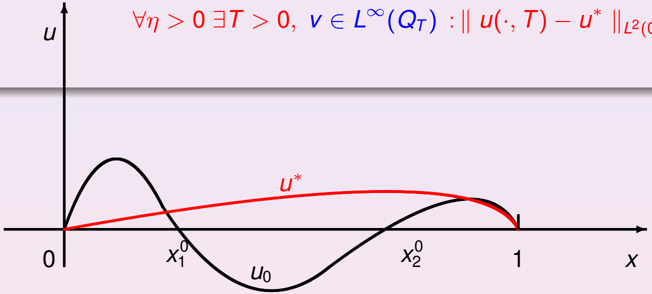

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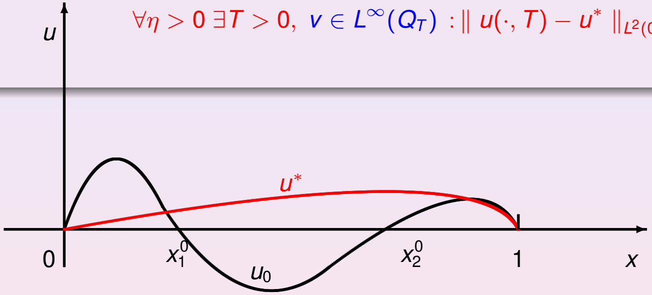

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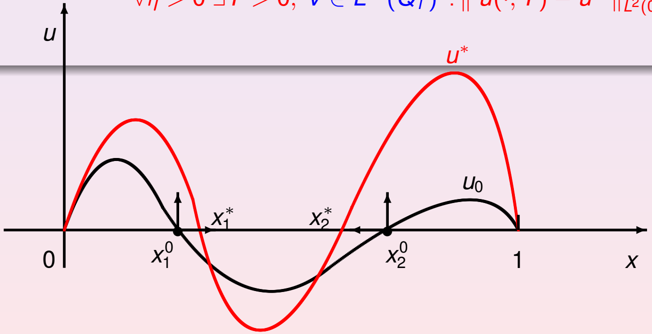
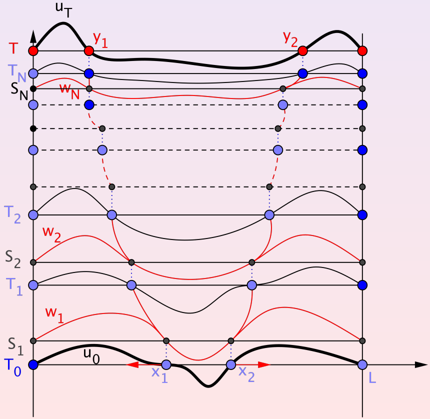


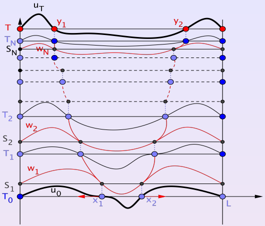
Control strategy

Given $N \in \mathbb{N}$ (N will be determined later) we consider the following partition of $[0, T]$ in $2N+1$ intervals:

$$[0, S_1] \cup [S_1, T_1] \cup \cdots \cup [T_{k-1}, S_k] \cup [S_k, T_k] \cup \cdots \cup [T_{N-1}, S_N] \cup [S_N, T_N] \cup [T_N, T]$$

$$v_1 \neq 0 \qquad 0 \qquad \cdots \qquad v_k \neq 0 \qquad 0 \qquad \cdots \qquad v_N \neq 0 \qquad 0 \qquad v_{N+1} \neq 0$$





Construction of the zero curves

- On $[S_k, T_k]$ $(1 \leqslant k \leqslant N)$ we use of the Cauchy datum $w_k \in C^{2+\theta}([0,1])$ in

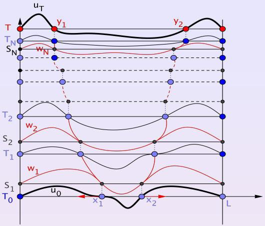
$$\begin{cases} w_t = w_{xx} + f(w), & \text{in } (0,1)\times[S_k,T_k], \\ w(0,t) = w(1,t) = 0, & t \in [S_k,T_k], \\ w\mid_{t=S_k} = w_k(x), & w_k''(x)\mid_{x=0,1} = 0, \end{cases}$$

  as a control parameter to be chosen to generate and to move the curves of sign change.

- The ℓ-th curve of sign change $(1 \leqslant \ell \leqslant n)$ is given given by solution ξ_ℓ^k

$$\begin{cases} \dot{\xi}_\ell^k(t) = -\frac{w_{xx}(\xi_\ell(t),t)}{w_x(\xi_\ell(t),t)}, & t \in [S_k,T_k] \\ \xi_\ell^k(S_k) = x_\ell^k \end{cases}$$

  where the x_ℓ^k's are the zeros of w_k and so $w(\xi_\ell^k(t),t) = 0$



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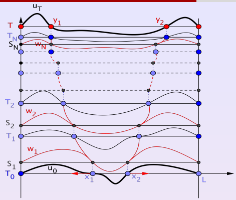


The control parameters w_k's will be chosen to move the curves of sign change in the following way

$$\begin{cases} \dot{\xi}_\ell(t) = -\frac{w_{xx}(\xi_\ell(t),t)}{w_x(\xi_\ell(t),t)}, & t \in [S_k, T_k] \\ \xi_\ell(S_k) = x_\ell^k, \end{cases} \qquad w(\xi_\ell^k(t),t) = 0 \Longrightarrow$$

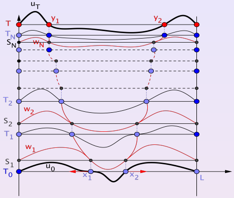
$$\Longrightarrow \dot{\xi}_\ell(S_k) = -\frac{w_{xx}(\xi_\ell(S_k), S_k)}{w_x(\xi_\ell(S_k), S_k)} = -\frac{w_k''(\xi_\ell(S_k))}{w_k'(\xi_\ell(S_k))} = \operatorname{sgn}(y_l - x_l^0)$$

- To fill the gaps between two successive $[S_k, T_k]$'s, on $[T_{k-1}, S_k]$ we construct the bilinear control v_k that steers the solution of

$$\begin{cases} u_t = u_{xx} + v_k(x,t)u + f(u), & \text{in } (0,1) \times [T_{k-1}, S_k], \\ u(0,t) = u(1,t) = 0, & t \in [T_{k-1}, S_k], \\ u\mid_{t=T_{k-1}} = u_{k-1} + r_{k-1}, \end{cases}$$

from $u_{k-1} + r_{k-1}$ to w_k, where u_{k-1} and w_k have the same points of sign change, and $\|r_{k-1}\|_{L^2(0,1)}$ is small.

Closing the loop



- The distance-from-target function satisfies the following estimate, for some $C_1, C_2 > 0$ and $0 < \theta < 1$,

$$0 \leq \sum_{\ell=1}^{n} |\xi_\ell^N(T_N) - y_\ell| \leq \sum_{\ell=1}^{n} |x_\ell^0 - y_\ell| + C_1 \sum_{k=1}^{N} \frac{1}{k^{1+\frac{\vartheta}{2}}} - C_2 \sum_{k=n+1}^{N} \frac{1}{k} \xrightarrow{N\to\infty} -\infty$$

- So the distances of each branch of the null set of the solution from its target points of sign change decreases at a linear-in-time rate while the error caused by the possible displacement of points already near their targets is negligible
- This ensures, by contradiction argument, that $\sum_{\ell=1}^{n} |\xi_\ell^N(T_N) - y_\ell| < \epsilon$ within a finite number of steps.

Outline

m-d radial case

$$\Omega = \{x \in \mathbb{R}^m : |x| = \sqrt{x_1^2 + \ldots + x_m^2} \le 1\}$$

$$\begin{cases} u_t = \Delta u + v(x,t)u + f(u) & \text{in } Q_T := \Omega \times (0,T) \\ u_{|\partial\Omega} = 0 & t \in (0,T) \\ u_{|t=0} = u_0 \end{cases}$$

u_0 and $v(\cdot, t)$ radial functions. Moreover, all possible hypersurface (lines) of sign change of u_0 are hyperspheres (circles) with center at the origin.

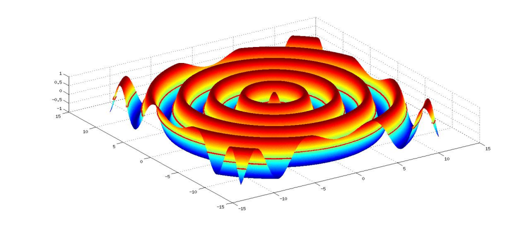


Figure: $u_0(x,y) = \cos(2\sqrt{x^2+y^2})$, initial state

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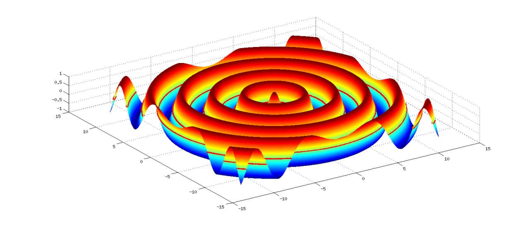

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Main results

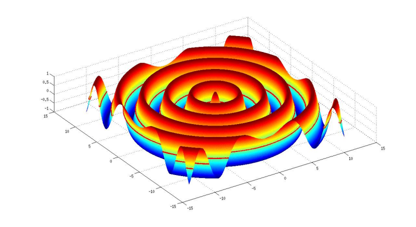
Theorem (G.F.)

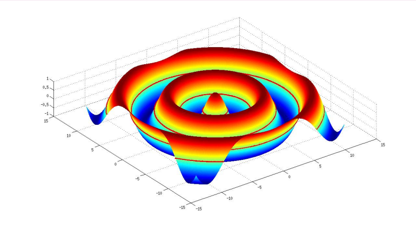
Let $u_0 \in H^2(\Omega)\bigcap H^1_0(\Omega)$. Assume that $u^* \in H^2(\Omega)\bigcap H^1_0(\Omega)$ has as many lines of sign change in the same order as $u_0(x)$. Then,

$$\forall \varepsilon > 0 \, \exists T > 0, v \in L^\infty(Q_T) \text{ such that } \| u(\cdot, T) - u^* \|_{L^2(\Omega)} < \varepsilon.$$

Corollary (G.F.)

The result of Theorem extends to the case when u^* has a lesser amount of lines of sign change which can be obtained by merging those of u_0.





u_0 and $v(\cdot, t)$ radial functions:

$$u_0(x) = z_0(r) \quad \text{and} \quad v(x,t) = V(r,t) \qquad \forall x \in \Omega,\ \forall t \in [0, T]$$

where $r = |x|$. Then,

$$\begin{cases} z_t = z_{rr} + \frac{m-1}{r} z_r + V(r,t)z + f(z) & \text{in} \quad (0,1)\times(0,T) \\ \lim\limits_{r\to 0^+} r^{\frac{m-2}{2}} z_r(0,t) = 0 = z(1,t) & t \in (0,T) \\ u_{|_{t=0}} = z_0. \end{cases}$$

z_0 has finitely many points of change of sign in $[0,1]$, that is, there exist points

$$0 = r_0^0 < r_1^0 < \cdots < r_n^0 < r_{n+1}^0 = 1$$

such that $\lim\limits_{r\to 0^+} r^{\frac{m-2}{2}} z_0'(r) = 0$ and

$$z_0(r) = 0 \quad \Longleftrightarrow \quad r = r_l^0, \quad l = 1, \dots, n+1$$

$$z_0(r) z_0(s) < 0, \quad \forall r \in \left(r_{l-1}^0, r_l^0\right), \forall s \in \left(r_l^0, r_{l+1}^0\right) \qquad l = 1, \dots, n.$$

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Outline

Semilinear degenerate problems

Initial states that change sign:

- 1-D degenerate equations (preprint with Carlo Nitsch and Cristina Trombetti).

$$\begin{cases} u_t - (a(x)u_x)_x = v(t,x)u + f(t,x,u) \quad \text{in } Q_T := (0,T)\times(-1,1) \\ \begin{cases} \begin{cases} \beta_0 u(t,-1) + \beta_1 a(-1)u_x(t,-1) = 0 & t\in(0,T) \\ \gamma_0\, u(t,1) + \gamma_1\, a(1)\, u_x(t,1) = 0 & t\in(0,T) \end{cases} & (\text{for } WDP) \\ a(x)u_x(t,x)|_{x=\pm 1} = 0 \quad t\in(0,T) & (\text{for } SDP) \end{cases} \\ u(0,x) = u_0(x) \quad x\in(-1,1)\,. \end{cases}$$

$$a \in C^0([-1,1]) \;:\; a(x) > 0 \,\forall\, x\in(-1,1),\; a(-1) = a(1) = 0$$

We distinguish two cases:

- ⋆ $\frac{1}{a} \in L^1(-1,1)$ (WDP);
- ⋆ $\frac{1}{a} \notin L^1(-1,1)$ (SDP) (e.g. $a(x) = 1 - x^2$, Budyko-Sellers climate model)

Let $u_0 \in H^1_a(-1,1)$ have *a finite number of points of sign change.*

Theorem (G.F., C. Nitsch, C. Trombetti, 2017)

Consider any $u^* \in H^1_a(-1,1)$ which has exactly as many points of sign change in the same order as u_0. Then,

$$\forall \eta > 0 \; \exists T > 0, \; \alpha \in L^\infty(Q_T) \; : \| \, u(\cdot, T) - u^* \, \|_{L^2(-1,1)} \leq \; \eta.$$

Corollary (G.F., C. Nitsch, C. Trombetti, 2017)

Consider any $u^* \in H^1_a(-1,1)$, whose amount of points of sign change is less than to the amount of such points for u_0 and these points are organized in any order of sign change. Then,

$$\forall \eta > 0 \; \exists T > 0, \; \alpha \in L^\infty(Q_T) \; : \| \, u(\cdot, T) - u^* \, \|_{L^2(-1,1)} \leq \; \eta.$$

Open problems

- Initial states that change sign: to investigate problems in higher space dimensions on domains with specific geometries
  - m-D degenerate case with radial symmetry
  - m-D non-degenerate case without radial symmetry and initial condition that change sign
- 1-D degenerate reaction-diffusion equations on networks
- To extend this approach to other nonlinear systems of parabolic type
  - Porous media equation (???)
  - The equations of fluid dynamics and swimming models: “Swimming models for incompressible Navier-Stokes equations”.

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Happy Birthday Professor Piermarco!!!

Thank you for your attention!