Maximum Principle and Sensitivity Relations in the Infinite Horizon Optimal Control

Hélène Frankowska

CNRS and UNIVERSITÉ PIERRE et MARIE CURIE

In collaboration with Piermarco Cannarsa

INdAM Workshop - New Trends in Control Theory and PDEs

On the occasion of the 60th birthday of Piermarco Cannarsa

July 3-7, 2017

Long Lasting Collaboration and Friendship

(1989) *Two characterisations of optimal trajectories for the Mayer problem*, IFAC symposium Nonlinear Control Systems Design

(1990) *Quelques characterisations des trajectoires optimales dans la théorie de contrôle*, Note CRAS

(1990) *Some characterizations of optimal trajectories in control theory*, Proceedings of 29th CDC Conference

(1991) *Some characterizations of optimal trajectories in control theory*, SICON

(1992) *Value function and optimality conditions for semilinear control problems*, AMO

(1996) *On the value function of semilinear optimal control problems of parabolic type, II*, AMO (Plant and Soil, Agroforestry Systems, Forest Ecology and Management, Field Crops Research)

Cannarsa P., Frankowska H. & Sinestrari C. (1998) *Properties of minimal time function in nonlinear control theory*, JMSEC

Cannarsa P., Frankowska H. & Sinestrari C. (2000) *Optimality conditions and synthesis for the minimum time problem*, SVA

(2006) *Interior sphere property of attainable sets and time optimal control problems*, ESAIM COCV

Cannarsa P., Frankowska H. & Marchini E. (2007) *Lipschitz continuity of optimal trajectories in deterministic optimal control*, View of ODE's

Cannarsa P., Frankowska H. & Marchini E. (2009) *Existence and Lipschitz regularity of solutions to Bolza problems in optimal control*, TAMS

Cannarsa P., Frankowska H. & Marchini E. (2009) *On Bolza optimal control problems with constraints*, DCDS, Series B

Cannarsa P., Da Prato G. & Frankowska H. (2010) *Invariant measures associated to degenerate elliptic operators*, Indiana University Mathematics Journal

Cannarsa P., Frankowska H. & Marchini E. (2013) *On optimal control problems with applications to systems with memory*, JEE

(2013) *Local regularity of the value function in optimal control*, Systems and Control Letters

(2014) *From pointwise to local regularity for solutions of Hamilton-Jacobi equations*, Calculus of Variations and PDEs

Cannarsa P., Frankowska H. & Scarinci T. (2014) *Sensitivity relations for the Mayer problem with differential inclusions*, Proceedings of 53rd CDC Conference

Cannarsa P., Frankowska H. & Scarinci T. (2015) *Sensitivity relations for the Mayer problem with differential inclusions*, ESAIM COCV

Cannarsa P., Frankowska H. & Scarinci T. (2016) *Second-order sensitivity relations and regularity of the value function for Mayer's problem in optimal control*, SICON

Cannarsa P., Da Prato G. & Frankowska H. (2016) *Invariance for quasi-dissipative systems in Banach spaces*, JMAA

(2017) *Value function, relaxation, and transversality conditions in infinite horizon optimal control*, JMAA

(2017) *Infinite horizon optimal control: transversality conditions and sensitivity relations*, Proceedings of ACC Conference

- Three common European projects (HCM, TMR, ITN Marie Curie)

- Two PhD students in co-tutelle (Teresa Scarinci and Vicenzo Basco)

- Also Marco Mazzola as a post-doctoral

- Many co-organised events, shared friends and co-authors, travels, diners, walks, discussions, confidences....

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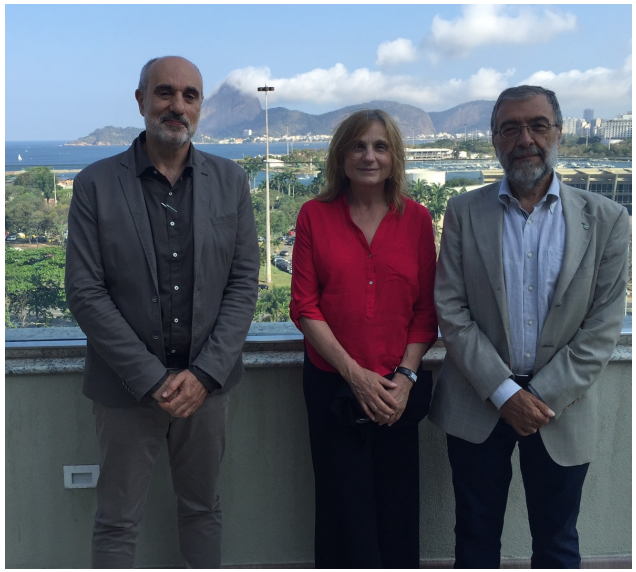

Infinite Horizon Optimal Control Problem

$$V(t_0, x_0) = \inf \int_{t_0}^{\infty} L(t, x(t), u(t))\, dt$$

over all trajectory-control pairs (x, u), subject to the state equation

$$\begin{cases} x'(t) = f(t, x(t), u(t)), \quad u(t) \in U(t) \quad \text{for a.e. } t \geq 0 \\ x(t_0) = x_0 \end{cases}$$

$x_0 \in \mathbb{R}^n$, $U : \mathbb{R}_+ \rightsquigarrow \mathbb{R}^m$ is a measurable set-valued map with closed $\neq \emptyset$ images, $L : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$.
Controls $u(t) \in U(t)$ are **Lebesgue measurable** selections.
L is bounded from below by a **function integrable** on $[0, \infty[$.
Thus V takes values in $(-\infty, +\infty]$.

A classical infinite horizon optimal control problem

$$W(x_0) = \text{minimize} \int_0^\infty e^{-\rho t} \ell(x(t), u(t))\, dt$$

over all trajectory-control pairs (x, u), subject to the state equation

$$\begin{cases} x'(t) = f(x(t), u(t)), \quad u(t) \in U \quad \text{for a.e. } t \geq 0 \\ x(0) = x_0 \end{cases}$$

controls $u(\cdot)$ are Lebesgue measurable, $\rho > 0$.

The literature addressing this problem deals with traditional questions of **existence** of optimal solutions, **regularity** of W, **necessary and sufficient** optimality conditions.

A. Seierstad and K. Sydsaeter. Optimal control theory with economic applications, 1986.

Hamilton-Jacobi Equation

$\ell \geq 0$. Under some technical assumptions W is the unique **bounded lower semicontinuous solution** with values in $\mathbb{R}_+$ of the Hamilton-Jacobi equation

$$\rho W(x) + \sup_{u \in U} \left(\langle \nabla W(x), f(x,u) \rangle - \ell(x,u) \right) = 0$$

in the following sense

$$\rho W(x) + \sup_{u \in U} \left(\langle p, f(x,u) \rangle - \ell(x,u) \right) = 0 \quad \forall\, p \in \partial^- W(x), \quad x \in \mathbb{R}^n$$

$\partial^- W(x)$ denotes the **subdifferential** of W at x.
HF and Plaskacz 1999, in the presence of **state constraints**.

If W is Bounded and Uniformly Continuous, then it is also the unique viscosity solution in the set of BUC functions
Soner 1986, in the presence of **state constraints**.

Necessary Optimality Condition: Maximum Principle

If $(\bar{x}, \bar{u})$ is optimal, then $\exists\ p_0 \in \{0, 1\}$ and a locally absolutely continuous $p : [0, \infty[\to \mathbb{R}^n$ with $(p_0, p) \neq 0$,
solving the **adjoint system**

$$-p'(t) = p(t) f_x(t, \bar{x}(t), \bar{u}(t)) - p_0 L_x(t, \bar{x}(t), \bar{u}(t)) \quad \text{for a.e.} \quad t \geq 0$$

and satisfying the **maximality condition**

$$\langle p(t), f(t, \bar{x}(t), \bar{u}(t)) \rangle - p_0 L(t, \bar{x}(t), \bar{u}(t)) =$$

$$\max_{u \in U(t)} (\langle p(t), f(t, \bar{x}(t), u) \rangle - p_0 L(t, \bar{x}(t), u)) \quad \text{for a.e.} \quad t \geq 0$$

If $p_0 = 0$ this maximum principle (MP) is called abnormal.

Transversality condition like $\lim_{t \to \infty} p(t) = 0$ is, in general, absent, cf. Halkin 1974.

Main Differences with the Finite Horizon Case

- The maximum principle may be **abnormal**
- Transversality conditions are not derived from the cost : some authors, under appropriate assumptions, obtain a **transversality condition** at infinity in the form

$$\lim_{t\to\infty} p(t) = 0 \quad \text{or} \quad \lim_{t\to\infty} \langle p(t), \bar{x}(t)\rangle = 0$$

However they are a consequence of the growth assumptions on f, L

Main difficulty behind : Restriction of an optimal solution to a finite time interval is no longer optimal

Some Approaches to Get Necessary Conditions

- To work with "finitely" optimal controls Halkin 1974; Carlson, Haurie 1987. This leads to finite horizon problems with an end-point constraint
- Penalization and dealing with limits of finite horizon problems Aseev and Kryazhimskiy, 2004
- Locally weakly overtaking optimal controls Aseev and Veliov, 2015
- An alternative for linear control systems by Aubin and Clarke, 1979 : duality theory on weighted Sobolev spaces $L^p(0,\infty;\mathbb{R}^n)$ with the measure $e^{-\rho t}dt$ and, more recently, for more general measures Pickenhain 2010; Tauchnitz 2015
- Transversality condition at the initial state $-p(0) \in \partial W(x_0)$ (generalized gradient of W at x_0), Aubin and Clarke, 1979 Followed by works of Michel 1982; Ye 1993; Sagara 2010

Dynamic Programming Principle

L is bounded from below by an **integrable function**, that is $L(t,x,u) \geq \alpha(t)$ for a.e. $t \geq 0$ and all x, u, where $\alpha : \mathbb{R}_+ \to \mathbb{R}$ is integrable on $[0, +\infty[$. Thus V takes values in $(-\infty, +\infty]$.

For every (t_0, x_0) with $V(t_0, x_0) < \infty$ the dynamic programming principle holds true: if $(\bar{x}, \bar{u})$ is optimal, then for every $T > t_0$,

$$V(t_0, x_0) = V(T, \bar{x}(T)) + \int_{t_0}^{T} L(t, \bar{x}(t), \bar{u}(t))\, dt$$

and for any other trajectory control pair (x, u)

$$V(t_0, x_0) \leq V(T, x(T)) + \int_{t_0}^{T} L(t, x(t), u(t))\, dt$$

Reduction to the Bolza Problem with Finite Horizon

Introducing $g_T(y) := V(T, y)$ we get, using the **dynamic programming principle**, the **Bolza** type problem

$$V^B(t_0, x_0) := \inf \left(g_T(x(T)) + \int_{t_0}^{T} L(t, x(t), u(t))\, dt \right)$$

over all trajectory-control pairs (x, u), subject to the state equation

$$\begin{cases} x'(t) = f(t, x(t), u(t)), \quad u(t) \in U(t) \quad \text{for a.e. } t \in [t_0, T] \\ x(t_0) = x_0 \end{cases}$$

g_T may be discontinuous and so the (MP) is not immediate.

Under assumptions (H) below, $V^B(s_0, y_0) = V(s_0, y_0)$ for all $s_0 \in [0, T]$, $y_0 \in \mathbb{R}^n$. Furthermore, if $(\bar{x}, \bar{u})$ is optimal for the infinite horizon problem at (t_0, x_0) then the restriction of $(\bar{x}, \bar{u})$ to $[t_0, T]$ is optimal for the above Bolza problem.

Assumptions (H)

i) $\exists$ locally integrable $c, \theta : \mathbb{R}_+ \to \mathbb{R}_+$ such that for a.e. $t \geq 0$

$$|f(t,x,u)| \leq c(t)|x| + \theta(t), \quad \forall\, x \in \mathbb{R}^n,\ u \in U(t);$$

ii) $\forall\, R > 0$, $\exists$ a locally integrable $c_R : \mathbb{R}_+ \to \mathbb{R}_+$ such that for a.e. $t \geq 0$, $\forall\, x, y \in B(0,R)$, $\forall\, u \in U(t)$

$$|f(t,x,u) - f(t,y,u)| + |L(t,x,u) - L(t,y,u)| \leq c_R(t)|x-y|;$$

iii) $\forall\, x \in \mathbb{R}^n$, $f(\cdot,x,\cdot)$, $L(\cdot,x,\cdot)$ are Lebesgue-Borel measurable ;

iv) $\exists$ a locally integrable $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ and a locally bounded nondecreasing $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that for a.e. $t \geq 0$,

$$L(t,x,u) \leq \beta(t)\phi(|x|), \quad \forall\, x \in \mathbb{R}^n,\ u \in U(t);$$

v) $U(\cdot)$ is Lebesgue measurable and has closed nonempty images;

vi) For a.e. $t \geq 0$, $\forall\, x \in \mathbb{R}^n$ the set $F(t,x)$ is closed and convex

$$F(t,x) := \{(f(t,x,u), L(t,x,u) + r) : u \in U(t) \text{ and } r \geq 0\}$$

Existence

For any $t_0 \in \mathbb{R}_+$, $x_0 \in \mathbb{R}^n$ such that $V(t_0, x_0) < +\infty$, a trajectory-control pair $(\bar{x}, \bar{u})$ is called **optimal** for the infinite horizon problem at (t_0, x_0) if for every trajectory-control pair (x, u) satisfying $x(t_0) = x_0$ we have

$$\int_{t_0}^{\infty} L(t, \bar{x}(t), \bar{u}(t))\, dt \leq \int_{t_0}^{\infty} L(t, x(t), u(t))\, dt$$

Proposition

Assume (H). Then V is lower semicontinuous and for every $(t_0, x_0) \in dom(V)$, there exists a trajectory-control pair $(\bar{x}, \bar{u})$ satisfying $V(t_0, x_0) = \int_{t_0}^{\infty} L(t, \bar{x}(t), \bar{u}(t))\, dt$.

Relaxation

Consider the **relaxed** infinite horizon problem

$$V^{rel}(t_0, x_0) = \inf \int_{t_0}^{\infty} \sum_{i=0}^{n} \lambda_i(t) L(t, x(t), u_i(t))\, dt$$

over all trajectory-control pairs of

$$\begin{cases} x'(t) = \sum_{i=0}^{n} \lambda_i(t) f(t, x(t), u_i(t)) \\ u_i(t) \in U(t),\ \lambda_i(t) \geq 0,\ \sum_{i=0}^{n} \lambda_i(t) = 1 \\ x(t_0) = x_0, \end{cases}$$

where $u_i(\cdot)$, $\lambda_i(\cdot)$ are Lebesgue measurable.
Clearly $V^{rel} \leq V$.

The above corresponds to the convexification of the set

$$F(t, x) := \{(f(t, x, u), L(t, x, u)) : u \in U(t)\}$$

Continuity of V^{rel} allows to omit (H) vi)

Theorem

Assume (H) i)-v) and that, for a.e. $t \geq 0$ and all $x \in \mathbb{R}^n$, the set

$$\{(f(t,x,u), L(t,x,u)) : u \in U(t)\}$$

is compact.
If for every $t \geq 0$, $V^{rel}(t,\cdot) : \mathbb{R}^n \to \mathbb{R}$ is continuous, then $V^{rel} = V$ on $\mathbb{R}_+ \times \mathbb{R}^n$. In particular, if a trajectory-control pair $(\bar{x}, \bar{u})$ is optimal, then it is also optimal for the relaxed problem.

The above assumption is verified if $U(t)$ is compact a.e. and $f(t,x,\cdot)$, $L(t,x,\cdot)$ are continuous. We introduce the **Hamiltonian**

$$H(t,x,p) := \sup_{u \in U(t)} \left(\langle p, f(t,x,u)\rangle - L(t,x,u)\right)$$

Generalized differentials and Limiting Normals

$\mathrm{hyp}(\varphi)$ - **hypograph** of $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$. For $x \in dom(\varphi)$

$$\partial^- \varphi(x) := \{p \mid \liminf_{y \to x} \frac{\varphi(y) - \varphi(x) - \langle p, y - x\rangle}{|y - x|} \geq 0\}$$

For $K \subset \mathbb{R}^n$ and $x \in K$

$$T_K(x) := \{v | \liminf_{h \to 0+} \frac{d_K(x + hv)}{h} = 0\}, \quad N_K^L(x) = \mathrm{Limsup}_{y \to_K x}[T_K(y)]^-$$

where $T_K(y)^-$ is the negative polar of $T_K(y)$.

Limiting superdifferential resp. **horizontal superdifferential:**

$$\partial^{L,+} \varphi(x) := \{p \mid (-p, 1) \in N^L_{hyp(\varphi)}(x, \varphi(x))\}$$

$$\partial^{\infty,+} \varphi(x) := \{p \mid (-p, 0) \in N^L_{hyp(\varphi)}(x, \varphi(x))\}$$

If φ is loc. Lipschitz, $\partial\varphi(x)$ - **generalized gradient** of φ at x.
For loc. Lipschitz $\psi : \mathbb{R}^n \to \mathbb{R}^n$, $\partial\psi(x)$ denotes the **generalized Jacobian** of ψ at x.

Maximum Principle for LSC Value Function

Theorem (normal MP with a sensitivity relation)

Let $(\bar{x}, \bar{u})$ *be optimal at* (t_0, x_0) *and* $\partial_x^- V(t_0, x_0) \neq \varnothing$.
If $f(t, \cdot, u)$ *and* $L(t, \cdot, u)$ *are differentiable, then*
$\forall\, p_0 \in \partial_x^- V(t_0, x_0)$ *the solution* $p(\cdot)$ *of the adjoint system*

$$-p'(t) = p(t) f_x(t, \bar{x}(t), \bar{u}(t)) - L_x(t, \bar{x}(t), \bar{u}(t)), \quad p(t_0) = -p_0$$

satisfies for a.e. $t \geq t_0$ *the* ***maximality condition***

$$\langle p(t), f(t, \bar{x}(t), \bar{u}(t)) \rangle - L(t, \bar{x}(t), \bar{u}(t)) = H(t, \bar{x}(t), p(t))$$

and the ***sensitivity relation***

$$-p(t) \in \partial_x^- V(t, \bar{x}(t)) \quad \forall\, t \geq t_0.$$

Maximum Principle for any $(t_0, x_0) \in dom(V)$

Theorem (MP without transversality condition)

Assume (H) and that $(\bar{x}, \bar{u})$ is optimal at (t_0, x_0). Then
*i) either **normal (MP)** holds: $\exists\, p(\cdot)$ solving the adjoint inclusion*

$$-p'(t) \in p(t)\partial_x f(t, \bar{x}(t), \bar{u}(t)) - \partial_x L(t, \bar{x}(t), \bar{u}(t)) \quad a.e. \quad t \geq t_0$$

and satisfying the maximality condition a.e. in $[t_0, +\infty[$
*ii) or **abnormal (MP)** holds: $\exists\, p(\cdot) \neq 0$ solving*

$$-p'(t) \in p(t)\partial_x f(t, \bar{x}(t), \bar{u}(t)) \quad a.e. \quad t \geq t_0,$$

and satisfying a.e. in $[t_0, +\infty[$ the abnormal maximality condition

$$\langle p(t), f(t, \bar{x}(t), \bar{u}(t))\rangle = \max_{u \in U(t)} \langle p(t), f(t, \bar{x}(t), u)\rangle.$$

Normality of the Maximum Principle

The infinite horizon problem is called **calm** with respect to the state variable at $(t_0, x_0) \in dom(V)$ if

$$\liminf_{y \to x_0} \frac{V(t_0, y) - V(t_0, x_0)}{|y - x_0|} > -\infty.$$

Theorem

Assume (H) $i) - v)$ and that the infinite horizon problem is calm with respect to the state variable at $(t_0, x_0) \in dom(V)$.
If a trajectory-control pair $(\bar{x}, \bar{u})$ is optimal at (t_0, x_0), then a normal (MP) holds true.

Maximum Principle with a Transversality Condition

Theorem

Let (H) i) − v) hold and $(\bar{x}, \bar{u})$ *be optimal at* (t_0, x_0). *If an upper semicontinuous function* $\Phi : \mathbb{R}^n \to \mathbb{R}$ *satisfies* $\Phi(\cdot) \leq V(t_0, \cdot)$ *on* $B(x_0, r)$ *for some* $r > 0$ *and* $\Phi(x_0) = V(t_0, x_0)$, *then*

i) either the normal (MP) holds true with the ***transversality condition***

$$-p(t_0) \in \partial^{L,+}\Phi(x_0);$$

ii) or the abnormal (MP) holds true with the ***transversality condition***

$$-p(t_0) \in \partial^{\infty,+}\Phi(x_0).$$

Transversality Condition and Sensitivity Relation

Theorem

Assume (H) $i)-v)$ and that $V(T,\cdot)$ is locally Lipschitz for all large $T>0$. Then for every $t\geq 0$, $V(t,\cdot)$ is locally Lipschitz with the local Lipschitz constant depending only on the magnitude of t. Moreover, if $(\bar{x},\bar{u})$ is optimal at some (t_0,x_0), then the normal (MP) holds true together with the sensitivity relations

$$-p(t_0)\in\partial_x V(t_0,x_0),\quad -p(t)\in\partial_x V(t,\bar{x}(t))\quad \text{for a.e. } t>t_0.$$

If c, θ, β are bounded and $F(t,x)$ are closed, then (MP) holds true with the adjoint system in the Hamiltonian form with p satisfying in addition the sensitivity relation

$$(H(t,\bar{x}(t),p(t)),-p(t))\in\partial V(t,\bar{x}(t))\quad \text{a.e.}$$

Second Order Jets

Let $\varphi\colon \mathbb{R}^n \to [-\infty,\infty]$ and $x \in dom(\varphi)$.
A pair $(q,Q) \in \mathbb{R}^n \times \mathbf{S}(\mathrm{n})$ is a **subjet** of φ at x if

$$\varphi(x) + \langle q, y-x\rangle + \frac{1}{2}\langle Q(y-x), y-x\rangle \leq \varphi(y) + o(|y-x|^2)$$

for some $\delta > 0$ and for all $y \in x + \delta B$. Then $q \in \partial^- \varphi(x)$.
The set of all subjets of φ at x is denoted by $J^{2,-}\varphi(x)$.

We assume next that $H \in C^{2,1}_{loc}$, that f, L are differentiable with respect to x and consider an optimal trajectory-control pain $(\bar{x},\bar{u})$ starting at (t_0,x_0).

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We assume next that $H \in C^{2,1}_{loc}$, that f, L are differentiable with respect to x and consider an optimal trajectory-control pain $(\bar{x}, \bar{u})$ starting at (t_0, x_0).

Riccati Equation

For a fixed $p_0 \in \partial_x^- V(t_0, x_0)$ let $\bar{p}(\cdot)$ solves the adjoint system with $\bar{p}(t_0) = -p_0$.
We already know that $-\bar{p}(t) \in \partial_x^- V(t, \bar{x}(t))$ for all $t \geq t_0$.

If for some $T > t_0$, $V(t, \cdot) \in C^2$ for all $t \in [t_0, T]$, then the Hessian $-V_{xx}(t, \bar{x}(t))$ solves the **matrix Riccati equation**:

$$\dot{R}(t) + H_{px}[t]R(t) + R(t)H_{xp}[t] + R(t)H_{pp}[t]R(t) + H_{xx}[t] = 0$$

where $H_{px}[t]$ abbreviates $H_{px}(t, \overline{x}(t), \overline{p}(t))$, and similarly for $H_{xp}[t], H_{pp}[t], H_{xx}[t]$.

Forward Propagation of Subjets

Theorem (corollary of Cannarsa, HF, Scarinci, SICON, 2016)

Assume $(p_0, R_0) \in J_x^{2,-} V(t_0, x_0)$ *for some* $R_0 \in \boldsymbol{S}(\mathrm{n})$.
If the solution R *of the matrix Ricatti equation*

$$\dot{R}(t) + H_{px}[t]R(t) + R(t)H_{xp}[t] + R(t)H_{pp}[t]R(t) + H_{xx}[t] = 0$$

with $R(t_0) = -R_0$ *is defined on* $[t_0, T]$, $T > t_0$, *then the following* ***second order sensitivity relation*** *holds true:*

$$(-\bar{p}(t), -R(t)) \in J_x^{2,-} V(t, \bar{x}(t)),\ \forall\, t \in [t_0, T].$$

Similar result is valid for backward propagation of second order superjets.

State Constraints

We request, in addition, that for a closed set $K \subset \mathbb{R}^n$ trajectories of the control system have to satisfy the state constraint

$$x(t) \in K, \quad \forall\, t \geq t_0$$

Assume that U is time independent and the following **inward pointing condition (IPC)** holds true:
$\exists\, \delta > 0$ such that $\forall\, x \in \partial K$ and $\forall\, u \in U$ with

$$\max_{n \in N_K^L(x) \cap S^{n-1}} \langle n, f(t,x,u) \rangle \geq 0$$

$$\exists\, w \in U \text{ satisfying } \max_{n \in N_K^L(x) \cap S^{n-1}} \langle n, f(t,x,w) - f(t,x,u) \rangle < -\delta$$

(Conditions developed together with M. Mazzola)
The usual H.M. Soner-type condition can not be applied in the non-autonomous case even for finite horizon problems : counterexamples to NFT theorems where given by A. Bressan.

Lipschitz Continuity of Value Function

Theorem

*Assume **(IPC)**, that (H) holds true with time independent c, θ, U, that f is continuous, $L(t,x,u) = e^{-\rho t}\ell(x,u)$, with $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ bounded, continuous and locally Lipschitz in x uniformly in u.*
Then for all $\rho > 0$ sufficiently large, V is locally Lipschitz and $V(t,\cdot)$ is locally Lipschitz uniformly in t.

The proof ot this result is so that it provides an estimate of ρ and of the local Lipschitz constants of V and $V(t,\cdot)$ from c, θ, c_R and the inward pointing condition.
This work is in progress with V. Basco and P. Cannarsa

Maximum Principle under State Constraints

Under the same assumptions, let $(\bar{x}, \bar{u})$ be optimal at $(t_0, x_0) \in \mathbb{R}_+ \times K$. Then there exists a locally absolutely continuous $p : [t_0, \infty[\to \mathbb{R}^n$ with $-p(t_0) \in \partial_x^{L,+} V(t_0, x_0)$, a positive Borel measure μ on $[t_0, \infty[$ and a Borel measurable

$$\nu(t) \in \Big(\operatorname{conv} N_K^L(\bar{x}(t))\Big) \cap B \qquad \mu - \text{a.e.} \quad t \geq t_0$$

such that for

$$\eta(t) := \int_{[t_0,t]} \nu(s) d\mu(s) \ \ \forall\, t > t_0 \quad \& \quad \eta(t_0) = 0,$$

$q(t) = p(t) + \eta(t)$ and for a.e. $t \geq t_0$, we have

$$(-\dot{p}(t), \dot{\bar{x}}(t)) \in \partial_{x,p} H(t, \bar{x}(t), q(t))$$

$$-q(t) \in \partial_x V(t, \bar{x}(t)), \quad (H(t, \bar{x}(t), q(t)), -q(t)) \in \partial\, V(t, \bar{x}(t))$$

grazie

I COMPLEANNI FANNO
BENE ALLA SALUTE
STUDI APPROFONDITI HANNO DIMOSTRATO
CHE LE PERSONE CHE FESTEGGIANO PIU'
COMPLEANNI VIVONO PIU' A LUNGO.

Local Lipschitz Continuity of $V(t,\cdot)$

Assume (H) with $c(t)\equiv c$, $\theta(t)\equiv\theta$, $c_R(t)\equiv\delta$ for all $R>0$ and

$$|L(t,y,u)-L(t,x,u)|\le k(t,|x|\vee|y|)|x-y|,\forall\, x,\, y\in\mathbb{R}^n,\, u\in U(t)$$

$k:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ is Lebesgue-Borel measurable, $k(t,\cdot)$ is $\nearrow$, and

$$\int_0^\infty e^{\delta t}k(t,(R+\theta t)e^{ct})dt<+\infty\quad\forall\, R\ge 0.$$

If $\mathrm{dom}\,(V)\neq\varnothing$, then V is locally Lipschitz on $\mathbb{R}_+\times\mathbb{R}^n$ and for all $t\ge 0$ and $R>0$

$$|V(t,x_2)-V(t,x_1)|\le e^{-\delta t}\,K_t(R)\,|x_2-x_1|\qquad\forall\, x_1,x_2\in B(0,R)$$

where for $M_t(\tau,R)=[R+\theta(\tau-t)]e^{c(\tau-t)}$

$$K_t(R):=\int_t^\infty e^{\delta\tau}k(\tau,M_t(\tau,R))\,d\tau$$

Remark. Less restrictive assumptions imply just continuity of V.

Behavior of the Co-state at ∞

Corollary

Under the same assumptions let $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$ *and* $(\bar{x}, \bar{u})$ *be any trajectory-control pair satisfying* $\bar{x}(t_0) = x_0$. *Then for all* $t \geq t_0$ *and* $x_1, x_2 \in B(\bar{x}(t), 1)$ *we have*

$$|V(t, x_2) - V(t, x_1)| \leq e^{-\delta t} K_{t_0}(1 + |x_0|)\,|x_2 - x_1|$$

Consequently, if $(\bar{x}, \bar{u})$ *is optimal and (MP) is augmented by the sensitivity relation*

$$-p(t) \in \partial_x V(t, \bar{x}(t)) \quad \text{a.e. } t > t_0$$

then $p(t) \to 0$ *exponentially when* $t \to \infty$.