Controllability properties of degenerate parabolic equations

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Dyrecta lab
Istituto di Ricerca

Outline

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Stochastic Flows

Let $X(\cdot, z)$ denote the unique solution to

$$\begin{cases} dX(t) = b(X(t))dt + \sigma(X(t))dW(t) & t \geq 0 \\ X(0) = z \in \mathbb{R}^d, \end{cases}$$

- $b : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^m)$ Lipschitz
- $W(t)$ $m-$dimensional Brownian motion

Consider the transition semigroup $P_t\varphi(z) = \mathbb{E}[\varphi(X(t,z))]$
Then $u(t,z) = P_t\varphi(z)$ is the solution of Kolmogorov equation

$$\begin{cases} u_t = \frac{1}{2}Tr[a(x)\nabla^2 u(x)] + \langle b(x), \nabla u(x)\rangle, & \text{in } (0,+\infty)\times\mathbb{R}^d \\ u(0,z) = \varphi(z) & x \in \mathbb{R}^d, \end{cases}$$

where $a(x) = \sigma(x)\sigma^*(x) \geq 0$

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Stochastic Invariance for Subset of $\mathbb{R}^d$

Denote the elliptic operator

$$Lu(x) := \frac{1}{2} Tr[a(x)\nabla^2 u(x)] + \langle b(x), \nabla u(x)\rangle$$

For any $\Omega \subset \mathbb{R}^d$ open set, let $\Gamma = \partial\Omega$ and

$$d_\Gamma(x) := \begin{cases} d(x;\Gamma) & \text{if } x \in \Omega \\ -d(x;\Gamma) & \text{if } x \in \Omega^c \end{cases}$$

the oriented distance from Γ.

A set $S \subset \mathbb{R}^d$ is invariant for X iff

$$z \in S \Rightarrow X(t,z) \in S \quad \mathbb{P}-\text{a.s. } \forall t \geq 0\,.$$

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Conditions for Invariance - References

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A. FRIEDMAN & M.A. PINSKY, *Asymptotic stability and spiraling properties of solutions of stochastic equations,* (1973)

J.P. AUBIN & G. DA PRATO, *Stochastic viability and invariance* (1990)

M. BARDI & P. GOATIN *Invariant sets for controlled degenerate diffusions: a viscosity solutions approach* (1999)

G. DA PRATO & H. FRANKOWSKA, *Stochastic viability for compact sets in terms of the distance function* (2001)

M. BARDI & R. JENSEN, *A geometric characterization of viable sets for controlled degenerate diffusions* (2002)

P. CANNARSA, G. DA PRATO & H. FRANKOWSKA, *Invariant measures associated to degenerate elliptic operators,* (2010)

P. CANNARSA & G. DA PRATO, *Stochastic Viability for regular closed sets in Hilbert spaces,* (2011)

P. CANNARSA & G. DA PRATO, *Invariance for stochastic reaction-diffusion equations,* (2012)

Characterization of Invariance

- Ω is invariant iff $\overline{\Omega}$ is so;
- the domain $\overline{\Omega}$ is invariant iff for all $x \in \Gamma$

$$
\begin{array}{ll}
(i) & Ld_\Gamma(x) \geq 0 \\
(ii) & \langle a(x)\nabla d_\Gamma(x), \nabla d_\Gamma(x)\rangle = 0
\end{array}
$$

- for any smooth function $\varphi : \overline{\Omega} \to \mathbb{R}$, the transition semigroup

$$u(x,t) = \mathbb{E}[\varphi(X(x,t))]$$

is the unique solution of the parabolic equation

$$
\begin{cases}
u_t = Lu & \text{in } \Omega \times (0,+\infty) \\
\langle a\nabla u, \nabla d_\Gamma\rangle = 0 & \text{on } \Gamma \times (0,+\infty) \\
u(x,0) = \varphi(x) & x \in \Omega\,,
\end{cases}
$$

i.e., L degenerates on Γ in the normal direction

Fluid Dynamics Models - Lin. Crocco & Prandl Eqs

Laminar flow ruled by the linearized Crocco's equation, $\Omega := (0,1) \times (0,L)$

$$\begin{cases} u_t + b(t,y)u_x - (a(y)u_y)_y + cu = f & (x,y,t) \in \Omega \times (0,T)\,, \\ u_y(x,0,t) = u(x,1,t) = 0 & (x,t) \in (0,L) \times (0,T)\,, \\ u(0,y,t) = u_1(y,t) & (y,t) \in (0,1) \times (0,T)\,, \\ u(x,y,0) = u_0(x,y) & (x,y) \in \Omega\,, \end{cases}$$

– f and u_1 depend on the incident velocity of the flow,

– coefficients a, b and c degenerate at the boundary

– double degeneracy of the diffusion coefficient, since

$$A(x,y) = \begin{pmatrix} 0 & 0 \\ 0 & a(0) \end{pmatrix}, \text{ with } a(0) = 0$$

Crocco's eq. simplifies the Prandl's equation for boundary layers (nonlinear and degenerate equation).

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Fluid Dynamics Models - Lin. Crocco Equation

References:

P. MARTINEZ; J.P. RAYMOND & J. VANCOSTENOBLE, *Regional null controllability of a linearized Crocco-type equation,* (2003)

Results for $1-$D degenerate equations

P. CANNARSA, P. MARTINEZ & J. VANCOSTENOBLE, *Persistent regional null controllability for a class of degenerate parabolic equations,* (2004)

P. CANNARSA, P. MARTINEZ & J. VANCOSTENOBLE, *Null controllability of degenerate heat equations,* (2005)

F. ALABAU-BOUSSOUIRA, P. CANNARSA & G. FRAGNELLI, *Carleman estimates for degenerate parabolic operators with applications to null controllability,* (2006)

P. CANNARSA, G. FRAGNELLI & J. VANCOSTENOBLE, *Regional controllability of semilinear degenerate parabolic equations in bounded domains,* (2006)

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P. CANNARSA, G. FRAGNELLI & J. VANCOSTENOBLE, *Regional controllability of semilinear degenerate parabolic equations in bounded domains,* (2006)

More References on $1-$D degenerate equations

P. CANNARSA & G. FRAGNELLI, *Null controllability of semilinear degenerate parabolic equations in bounded domains,* (2006)

P. CANNARSA, G. FRAGNELLI & D. ROCCHETTI, *Null controllability of degenerate parabolic operators with drift,* (2007)

P. CANNARSA, P. MARTINEZ & J. VANCOSTENOBLE, *Carleman estimates for a class of degenerate parabolic operators,* (2008)

P. CANNARSA, G. FRAGNELLI & D. ROCCHETTI, *Controllability results for a class of one-dimensional degenerate parabolic problems in nondivergence form,* (2008)

P. CANNARSA & L. DE TERESA, *Controllability of 1-D coupled degenerate parabolic equations,* (2009)

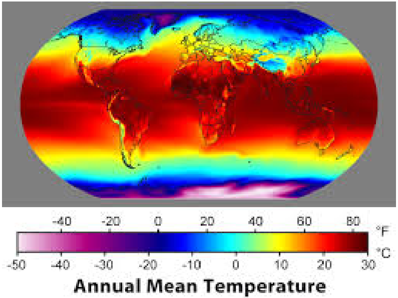
P. CANNARSA, J. TORT & M. YAMAMOTO, *Unique continuation and approximate controllability for a degenerate parabolic equation,* (2012)

M. GUEYE, *Exact Boundary Controllability of 1-D Parabolic and Hyperbolic Degenerate Equations,* (2014)

Budyko-Sellers climate models

Heat-balance equation for the sea-level averaged temperature

$$\begin{cases} cu_t - (k(1-x^2)u_x)_x = S_0 s(x)\alpha(x,u) - I(u)\,, & \text{in } (-1,1)\times(0,T)\,, \\ (1-x^2)u_x(t,x) = 0 & (x,t)\in\{-1,1\}\times(0,T)\,, \\ u(0,x) = u_0(x) & x\in(-1,1)\,, \end{cases}$$



c thermal capacity of the Earth,
k horizontal thermal conductivity,
S_0 solar constant, $s(x)$ normalized distribution of solar input,
α the coalbedo
$I(u)$ the outgoing infrared radiation (radiation emitted by the Earth)

Budyko-Sellers climate models - References

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References:

P. CANNARSA & A.Y. KHAPALOV, *Multiplicative controllability for reaction-diffusion equations with target states admitting finitely many changes of sign,* (2010)

P. CANNARSA & G. FLORIDIA, *Approximate controllability for linear degenerate parabolic problems with bilinear control,* (2011)

P. CANNARSA & G. FLORIDIA, *Approximate multiplicative controllability for degenerate parabolic problems with Robin boundary conditions* (2011)

P. CANNARSA, G. FLORIDIA & A.Y. KHAPALOV, *Multiplicative controllability for semilinear reaction-diffusion equations with finitely many changes of sign*, (2017)

More Examples...

– **Fleming-Viot diffusion process in population genetics:**

Consider the equation

$$u_t - Tr[A(x)\nabla^2 u] = g$$

in $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_i < 1,\ x_1 + x_2 \leq 1\}$, with

$$A(x_1, x_2) = \begin{pmatrix} x_1(1 - x_1) & -x_1 x_2 \\ -x_1 x_2 & x_2(1 - x_2) \end{pmatrix}$$

and A degenerates on $\partial\Omega$, indeed

$$det(A) = x_1 x_2 (1 - x_1 - x_2) = 0 \quad \text{on } \partial\Omega\,.$$

... and More Challenges

- in **mathematical finance**, the Black-Scholes equation

$$V_t + \frac{1}{2}\sigma^2 x^2 V_{xx} + rxV_x - rV = 0\,,$$

where V is the price of the option as a function of stock price x and time t, r is the risk-free interest rate, and σ is the volatility of the stock;

- **Porous Media Equation**, $m > 0$,

$$u_t = \Delta(u^m) = \nabla\cdot(c(u)\nabla u)$$

degenerates where $u = 0$ for $m > 1$;
Reference: J.M. CORON, J.I. DIAZ, A. DRICI & T. MINGAZZINI, *Global null controllability of the 1-dimensional nonlinear slow diffusion equation,* (2013)

- the $p-$**laplacian equation**

$$u_t = \nabla\cdot(|\nabla u|^{p-2}\nabla u)$$

is degenerate for $p > 2$ on $\{\nabla u = 0\}$

Outline

Unique Continuation & Approximate Controllability of Degenerate Parabolic Operators

joint work with Piermarco Cannarsa

Given $a : \mathbb{R} \to \mathbb{R}$ such that $a(0) = 0$ and $a > 0$ otherwise, study controllability properties of the parabolic equations

$$\begin{cases} y_t - (a(x) y_x)_x = 0\,, & \text{in } \; Q := (0,1) \times (0,T)\,, \\ y(0,t) = u(t)\,, & t \in (0,T)\,, \\ y(1,t) = 0\,, & t \in (0,T)\,, \\ y(x,0) = y_0(x)\,, & x \in (0,1)\,, \end{cases} \tag{BD.BC}$$

and, for some open $\omega \subset (-1, 0)$,

$$\begin{cases} y_t - (a(x) y_x)_x = \chi_\omega u\,, & \text{in } \tilde{Q} := (-1,1) \times (0,T)\,, \\ y(-1,t) = 0\,, & t \in (0,T)\,, \\ y(1,t) = 0\,, & t \in (0,T)\,, \\ y(x,0) = y_0(x)\,, & x \in (-1,1)\,, \end{cases} \tag{ID.DC}$$

Assumptions on the Degenerate Diffusion Coefficient

Case of Boundary Degeneracy & Boundary Control

$$\begin{cases} y_t - (a(x)y_x)_x = 0\,, & \text{in } \ Q := (0,1)\times(0,T)\,, \\ y(0,t) = u(t)\,, & t\in(0,T)\,, \\ y(1,t) = 0\,, & t\in(0,T)\,, \\ y(x,0) = y_0(x)\,, & x\in(0,1)\,, \end{cases}$$

where

H1 $a \in C([0,1]) \cap C^1((0,1])$, $a(0) = 0$, $a(x) > 0$ otherwise;

H2 There exist $\gamma, K \in (0,1)$ such that

$$\liminf_{x\to 0^+} \frac{xa'(x)}{a(x)} = \gamma\,, \qquad \limsup_{x\to 0^+} \frac{xa'(x)}{a(x)} = K\,.$$

(global version of H2: There exist $\gamma, K \in (0,1)$ such that

$$\gamma a(x) \le xa'(x) \le Ka(x) \text{ for all } x \in [0,1])$$

Main Results - Unique Continuation

Theorem (Unique continuation for (BD.BC))

$L^*v = v_t + (av_x)_x \quad in \quad Q$, and
$v \in L^2(0,T;D(A)) \cap H^1(0,T,L^2(0,1))$ such that

$$v(0,t) = (av_x)(0,t) = 0\,.$$

If $\quad L^*v \equiv 0 \quad in \quad Q, \;$ then $\quad v \equiv 0 \quad in \quad Q.$

Theorem (Unique continuation for (ID.DC))

$P^*v = v_t + (a(x)v_x)_x \quad in \quad \tilde{Q}$, and
$\tilde{v} \in H^1(0,T;L^2(-1,1)) \cap L^2(0,T;D(A_1))$ such that

$$\tilde{v} = 0 \quad in \;\; \omega \times (0,T)\,.$$

If $\quad P^*\tilde{v} = 0 \quad in \quad \tilde{Q}, \;$ then $\quad \tilde{v} = 0 \quad in \quad \tilde{Q}.$

Main Results - Approximate Controllability of (BD.BC)

Theorem

For all $y_0 \in L^2(0,1)$, $y_T \in L^2(0,1)$ and all $\varepsilon > 0$ there exists $u \in H_0^1(0,T)$ such that the solution y_u to

$$\begin{cases} y_t - (ay_x)_x = 0 & (x,t) \ \text{in} \ \ Q\,, \\ y(0,t) = u(t) & t \in (0,T)\ , \\ y(1,t) = 0 & t \in (0,T)\ , \\ y(x,0) = y_0(x) & x \in (0,1)\ , \end{cases}$$

satisfies

$$\|y_u(T) - y_T\|_{L^2(0,1)} \leq \varepsilon\,.$$

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For all $y_0 \in L^2(-1,1)$, $y_T \in L^2(-1,1)$ *and all* $\varepsilon > 0$ *there exists* $u \in L^2(\tilde{Q})$ *such that the solution* y_u *of problem*

$$\begin{cases} y_t - (a\, y_x)_x = \chi_\omega u & \text{in } (-1,1) \times (0,T)\,, \\ y(-1,t) = 0 = y(1,t) & t \in (0,T)\,, \\ y(x,0) = y_0(x) & x \in (-1,1)\,, \end{cases}$$

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Carleman Estimate with suitable space weight

Proof of the Unique Continuation based on new *Carleman estimate*, combining techniques from

F. ALABAU-BOUSSOUIRA, P. CANNARSA & G. FRAGNELLI, *Carleman estimates for degenerate parabolic operators with applications to null controllability,* (2006)

P. CANNARSA, J. TORT & M. YAMAMOTO, *Unique continuation and approximate controllability for a degenerate parabolic equation,* (2012)

Remark: in toy model $a(x) = x^\alpha$, $x \in (0,1)$, $\alpha \in (0,1)$

for AC: spatial weight $p(x) = -x^\beta$, for some $\beta \in (1-\alpha, 1-\alpha/2)$

for NC: spatial weight $p(x) = \frac{2-x^{2-\alpha}}{(2-\alpha)^2}$

from P. CANNARSA, P. MARTINEZ & J. VANCOSTENOBLE, *Carleman estimates for a class of degenerate parabolic operators,* (2008)

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