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Control and optimization of PDEs with degeneration

New Trends in Control Theory and PDEs

In the honor of Piermarco's 60th birthday

Günter Leugering

joint work with:

P. Cannarsa and F. Alabau (observability of the wave equation)

E. Casas and P.I. Kogut (weighted p-Laplace: control in the coefficients)

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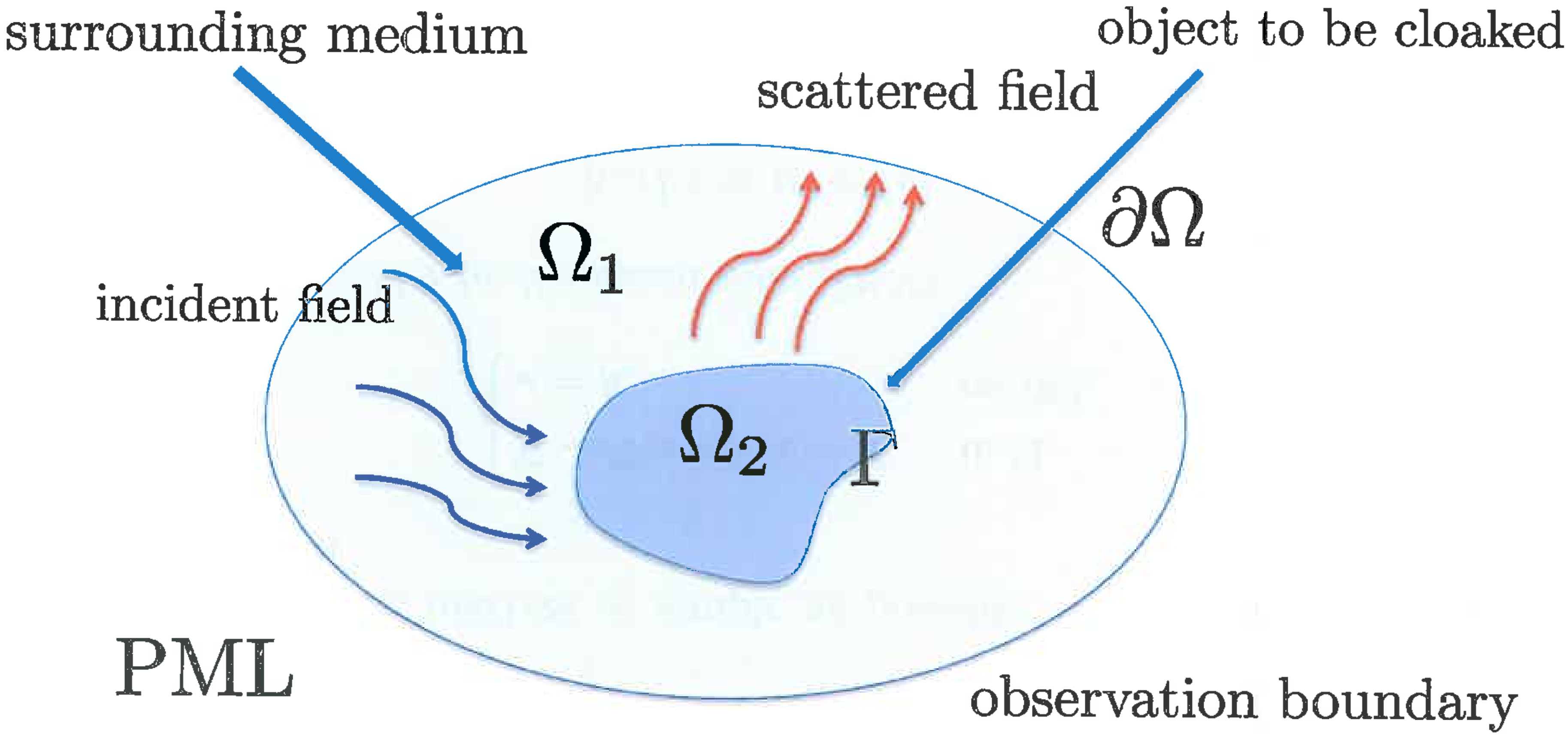
Motivation: Cloaking

Not seeing an object is equivalent to non-observability



In order to keep matters as simple as possible, we consider the following classical problem

$$\begin{cases} \nabla \cdot \sigma \nabla u + k^2 u = f, & \text{in } \Omega, \\ u = h, & \text{on } \partial\Omega. \end{cases}$$

We have the *Dirichlet-to-Neumann map* (DtN)

$$\Lambda_\sigma(h) := \nu \cdot \sigma \nabla u|_{\partial\Omega}.$$

$$\sigma(x) = \begin{cases} \sigma_1(x) & x \in \Omega \setminus \bar{D} \\ \sigma_2(x) & x \in D \end{cases}$$

Given

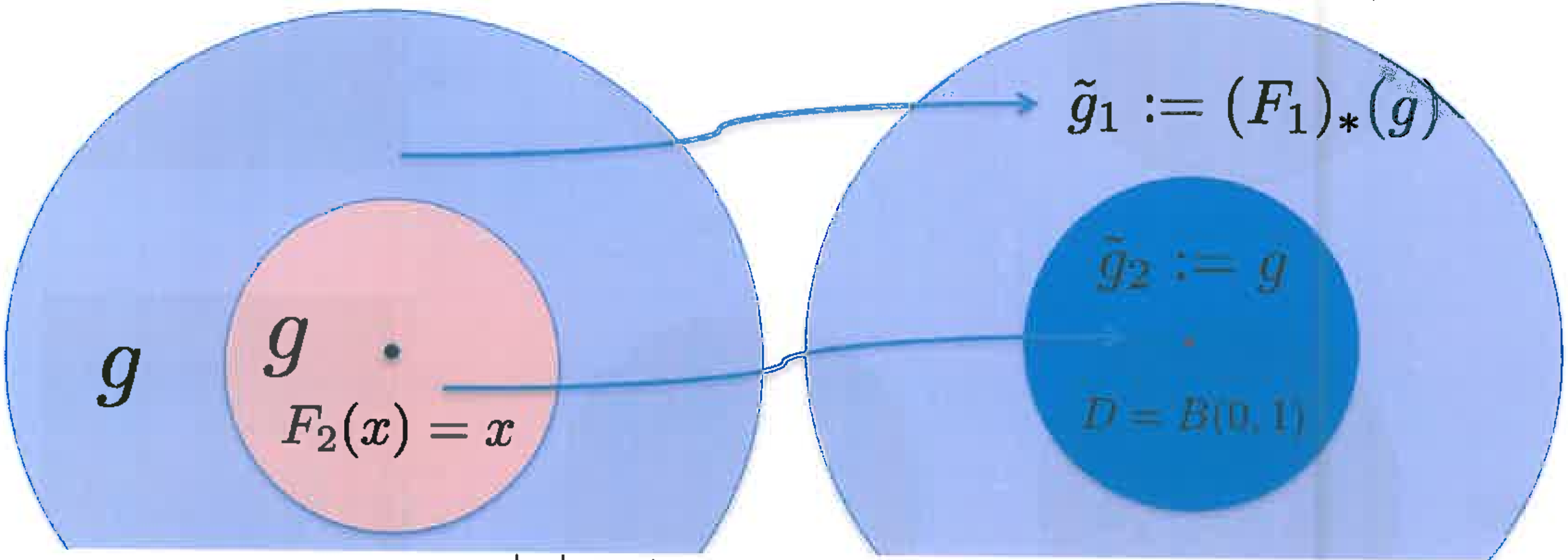
$$(u|_{\partial\Omega} = h, \Lambda_\sigma(h))$$

find σ_2 in D!

Standard setup: Change of coordinates

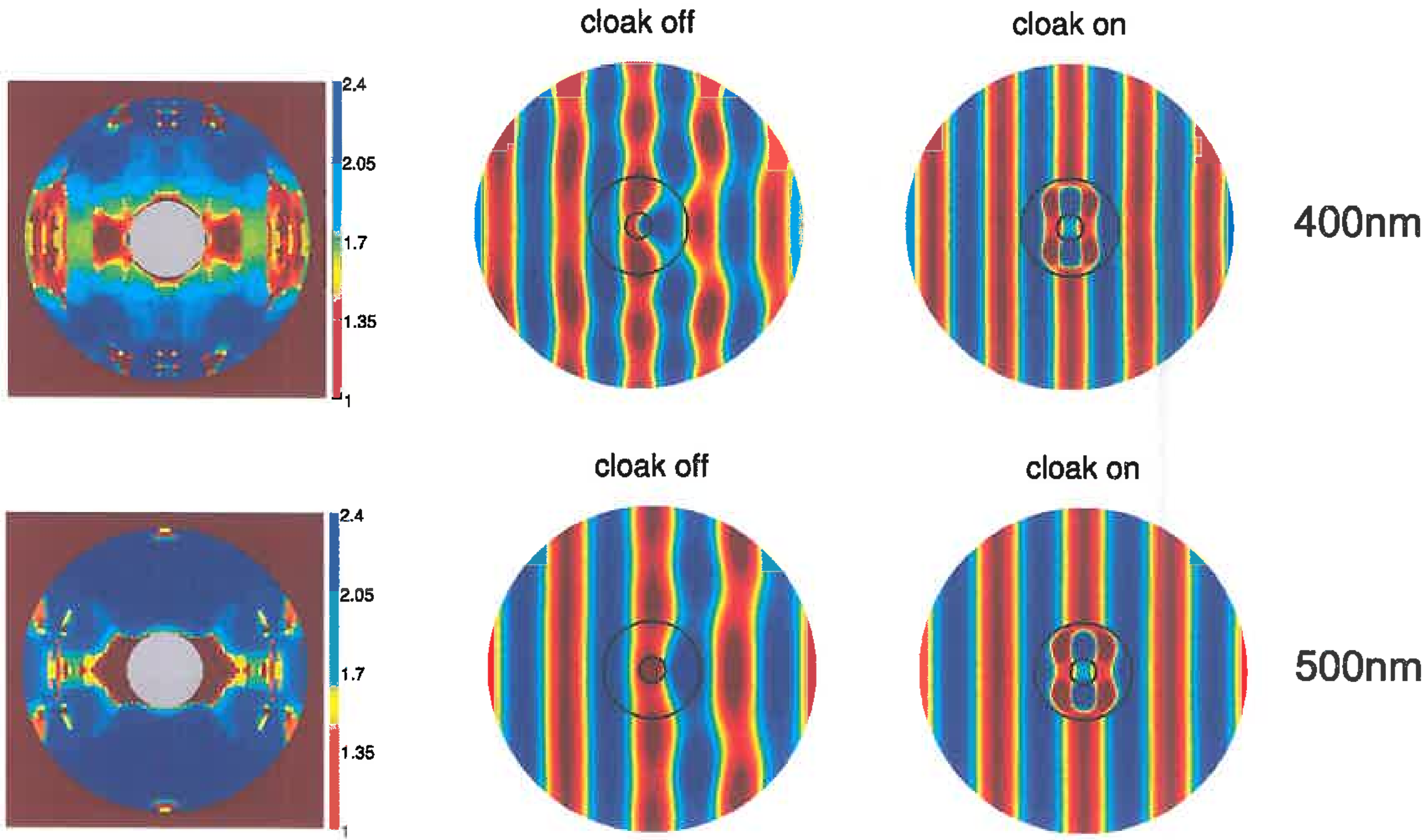
$N = N_1 \cup \Sigma \cup N_2 \quad N_1 = \Omega \setminus \bar{D} \quad N_2 = D \qquad F_1(x) = \left(\frac{|x|}{2} + 1\right)\frac{x}{|x|}$

$M := M_1 \cup M_2 \qquad M_1 := \Omega = B(0,2),\ M_2 = D = B(0,1),$



$$\tilde{\sigma}(x) = 2(I - P(x)) + 2\left(\frac{|x|-1}{|x|}\right)^2 P(x), 1 < |x| < 2, P(x) = \frac{xx^t}{|x|^2}$$

This transformation leads to a Riemann-metric with ***quadratic degeneration***!!
See the work of G. Uhlman and M. Latassas


cloak off
cloak on
2.4
2.05
1.7
1.35
1
400nm
cloak off
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1
500nm

Observability of degenerate 1-d wave equations

Joint work with P.Cannarsa and F. Alabau (SICON 2017)

We study the controllability and observability for degenerate wave equations of the form

$$u_{tt} - \left(a(x)u_x\right)_x = 0 \quad \text{in }]0,\infty[\times]0,1[, \tag{1}$$

where a is positive on $]0,1]$ but vanishes at zero.The degeneracy of (1) at $x = 0$ is measured by the parameter μ_a defined by

$$\mu_a := \sup_{0<x\leqslant 1} \frac{x|a'(x)|}{a(x)}, \tag{2}$$

and one says that (1) degenerates weakly if $\mu_a \in [0,1[$, strongly if $\mu_a > 1$. Here we assume $\mu_a < 2$

More precisely, let $a \in \mathcal{C}([0,1]) \cap \mathcal{C}^1(]0,1])$ be a function satisfying the following assumptions:

$$\begin{cases} (i) & a(x) > 0 \;\; \forall x \in]0,1]\,, \quad a(0) = 0\,, \\ (ii) & \mu_a := \sup_{0<x\leqslant 1} \dfrac{x|a'(x)|}{a(x)} < 2\,, \quad \text{and} \\ (iii) & a \in \mathcal{C}^{[\mu_a]}([0,1]), \end{cases}$$

where $[\cdot]$ stands for the integer part.

We now introduce some weighted Sobolev spaces that are naturally associated with degenerate operators. We denote by $V_a^1(0,1)$ the space of all functions $u \in L^2(0,1)$ such that

$$\begin{cases} (i) & u \text{ is locally absolutely continuous in }]0,1], \text{ and} \\ (iii) & \sqrt{a}u_x \in L^2(0,1). \end{cases}$$

It is easy to see that $V_a^1(0,1)$ is an Hilbert space with the scalar product

$$\langle u, v\rangle_{1,a} = \int_0^1 \big(a(x)u'(x)v'(x) + u(x)v(x)\big)\,dx\,, \qquad \forall\, u, v \in V_a^1(0,1)$$

and associated norm

$$\|u\|_{1,a} = \left\{\int_0^1 \big(a(x)|u'(x)|^2 + |u(x)|^2\big)\,dx\right\}^{\frac{1}{2}}, \qquad \forall\, u \in V_a^1(0,1)\,.$$

Let us also set

$$|u|_{1,a} = \left\{ \int_0^1 a(x)\,|u'(x)|^2 dx \right\}^{\frac{1}{2}} \qquad \forall\, u \in V_a^1(0,1).$$

Actually, $|\cdot|_{1,a}$ is an equivalent norm on the closed subspace of $V_{a,0}^1(0,1)$ defined as

$$V_{a,0}^1(0,1) = \{u \in V_a^1(0,1) \ : \ u(1) = 0\}.$$

This fact is a simple consequence of the following version of Poincaré's inequality for a with (2):

$$\|u\|_{L^2(0,1)}^2 \leqslant C_a\, |u|_{1,a}^2 \qquad \forall\, u \in V_{a,0}^1(0,1),$$

The following are examples of functions a satisfying assumption (2). Let $\theta \in]0,2[$ be given. Define

$$a(x) = x^{\theta} \qquad \forall x \in [0,1].$$

In this case, we have

$$\|u\|^2_{L^2(0,1)} \leqslant \min\left\{4, \frac{1}{2-\theta}\right\} |u|^2_{1,a} \qquad \forall u \in V^1_{a,0}(0,1).$$

Next, we define

$$V^2_a(0,1) = \left\{u \in V^1_a(0,1) \ : \ au' \in H^1(0,1)\right\},$$

where $H^1(0,1)$ denotes the classical Sobolev space of all functions $u \in L^2(0,1)$ such that $u' \in L^2(0,1)$. Notice that, if $u \in V^2_a(0,1)$, then au' is continuous on $[0,1]$.

Given a satisfying assumptions (2), let $\mu_a \in [0,2[$ be the constant in assumption (ii). Consider the degenerate wave equation

$$u_{tt} - \big(a(x)u_x\big)_x = 0 \quad \text{in } \ ]0,\infty[\times]0,1[\tag{1}$$

with

$$\begin{cases} \text{boundary conditions } \ u(t,1) = 0 \ \text{ and } \begin{cases} u(t,0) = 0 & \mu_a \in [0,1[\\ \lim_{x\downarrow 0} a(x)\,u_x(t,x) = 0 & \mu_a \in [1,2[\end{cases} \\ \text{initial conditions } \begin{cases} u(0,x) = u_0(x) \\ u_t(0,x) = u_1(x) \end{cases} \end{cases}$$

We recall that, since equation (1) is degenerate, different boundary conditions have to be imposed at $x = 0$ depending on whether we are interested in:

i.) the *weakly degenerate* case $\mu_a \in [0, 1[$, where, we have that the Dirichlet boundary condition $u(t, 0) = 0$ makes sense for any solution, and

ii.) the *strongly degenerate* case $\mu_a \in [1, 2[$, where, we have that the Neumann boundary condition $\lim_{x \downarrow 0} a(x)\, u_x(t, x) = 0$ is automatically satisfied by any classical solution.

In order to express the above boundary conditions in functional settings, we define $H^1_a(0,1)$ to be the closed subspace of $V^1_{a,0}(0,1)$ which consists of all $u \in V^1_{a,0}(0,1)$ satisfying $u(0)=0$ when $\mu_a \in [0,1[$. We also set

$$H^2_a(0,1) = V^2_a(0,1) \cap H^1_a(0,1).$$

Observe that all functions $u \in H^2_a(0,1)$ satisfy homogeneous boundary conditions at both $x=0$ and $x=1$. Such conditions are of Dirichlet type in the weakly degenerate case, whereas they are of Neumann/Dirichlet type at $x=0$ and $x=1$, respectively, when $\mu_a \in [1,2[$.

For any mild solution u of (1) we have that $u_x(\cdot,1) \in L^2(0,T)$ for every $T \geqslant 0$ and

$$a(1)\int_0^T u_x^2(t,1)\,dt \leqslant \left(6T + \frac{1}{\min\{1,a(1)\}}\right)E_u(0).$$

Moreover,

$$a(1)\int_0^T u_x^2(t,1)\,dt = \int_0^T\int_0^1 \Big\{u_t^2(t,x) + \big(a(x) - xa'(x)\big)u_x^2(t,x)\Big\}dtdx \\ + 2\Big[\int_0^1 xu_x(t,x)u_t(t,x)dx\Big]_{t=0}^{t=T}.$$

For any mild solution u of (1) we have that, for every $T \geqslant 0$,

$$\int_0^T\int_0^1 \{a(x)u_x^2(t,x) - u_t^2(t,x)\}dtdx + \Big[\int_0^1 u(t,x)u_t(t,x)dx\Big]_{t=0}^{t=T} = 0\,.$$

Assume (2) and let u be the mild solution of (1). Then, for every $T \geqslant 0$,

$$a(1) \int_0^T u_x^2(t,1)\,dt \geqslant \left\{(2-\mu_a)T - \frac{4}{\min\{1, a(1)\}} - 2\,\mu_a \sqrt{C_a}\right\} E_u(0)\,,$$

where C_a is the constant in the Poincare inequality.

We recall that (1) is said to be *observable* (via the normal derivative at $x=1$) *in time* $T > 0$ if there exists a constant $C > 0$ such that for any $(u_0, u_1) \in H_a^1(0,1) \times L^2(0,1)$ the mild solution of (1) satisfies

$$\int_0^T u_x^2(t,1)\,dt \geqslant C\,E_u(0)\,.$$

Any constant satisfying this is called an *observability constant* for (1) in time T. The supremum of all observability constants for (1) is denoted by C_T

Equivalently, (1) is observable if

$$C_T = \inf_{(u_0,u_1)\neq(0,0)} \frac{\int_0^T u_x^2(t,1)\,dt}{E_u(0)} > 0\,.$$

The inverse $c_T = 1/C_T$ is sometimes called the cost of observability (or the cost of control) in time T.

Corollay:
Assume (2). Then (1) is observable in time T provided that

$$T > T_a := \frac{1}{(2-\mu_a)} \left(\frac{4}{\min\{1, a(1)\}} + 2\,\mu_a \sqrt{C_a} \right)\,,$$

where C_a is defined as above. In this case

$$C_T \geqslant \frac{1}{a(1)} \Big\{ (2-\mu_a)T - \frac{4}{\min\{1, a(1)\}} - 2\,\mu_a \sqrt{C_a} \Big\}.$$

Let a be the power x^θ. Then we can apply the above to conclude that, defining

$$T_\theta = \frac{1}{2-\theta}\left(4 + 2\theta \min\left\{2, \frac{1}{\sqrt{2-\theta}}\right\}\right),$$

we have that

$$C_T \geqslant (2-\theta)(T - T_\theta) \qquad \forall\, T \geqslant T_\theta\,.$$

Observe that $T_\theta \to 2$ as $\theta \downarrow 0$, which coincides with the classical observability time for the wave equation.

We shall see that boundary observability is no longer true when the constant μ_a in (2) is greater than or equal to 2 and that, for $\mu_a < 2$, the controllability time blows up as $\mu_a \uparrow 2$. We discuss two examples with power-like coefficients.

1.) Given $T > 0$, consider the problem

$$\begin{cases} u_{tt} - \left(x^2 u_x\right)_x = 0 &]0,T[\times]0,1[\\ \text{boundary conditions: } u(t,1) = 0 \text{ and } \lim_{x\downarrow 0} x^2 u_x(t,x) = 0 & 0 < t < T \\ \text{initial conditions: } \begin{cases} u(0,x) = u_0(x) \\ u_t(0,x) = u_1(x) \end{cases} & x \in]0,1[\,, \end{cases}$$

where u_0 and u_1 are smooth functions with compact support in $]0,1[$, not identically zero.

Observe that the so-called Liouville transform

$$u(t,x) = \frac{1}{\sqrt{x}} v\left(t, \log\frac{1}{x}\right)$$

turns problem (1) into

$$\begin{cases} v_{tt} - v_{yy} + \dfrac{1}{4}v = 0 & \text{in } \;]0,T[\times]0,\infty[\\ v(t,0) = 0 & 0 < t < T \\ \text{initial conditions: } \begin{cases} v(0,y) = e^{-y/2}u_0(e^{-y}) := v_0(y) \\ v_t(0,y) = e^{-y/2}u_1(e^{-y}) := v_1(y)\,. \end{cases} & y \in]0,\infty[\,. \end{cases}$$

Since the support of the initial data propagates at finite speed, the normal derivative $v_y(\cdot,0)$ of the solution may well be identically zero on $[0,T]$ when the support of v_0 and v_1 is sufficiently far from $y=0$. Consequently, problem (1) is not observable on $[0,T]$ via the normal derivative $u_x(\cdot,1)$.

$$\varphi(x) := \int_x^1 \frac{ds}{s^{\theta/2}} = \frac{2(x^{1-\theta/2}-1)}{\theta-2}, \; \psi(y) = \Big(\frac{2}{2+(\theta-2)y}\Big)^{\frac{2}{\theta-2}}$$

$$u(t,x) := \frac{1}{x^{\theta/4}} v\big(t,\varphi(x)\big)$$

Then the problem transforms into

$$\begin{cases} v_{tt} - v_{yy} + \dfrac{c(\theta)}{[2+(\theta-2)y]^2}\, v = 0 & \text{in }]0,T[\times]0,\infty[\\ v(t,0) = 0 & 0 < t < T \\ \text{initial conditions: } \begin{cases} v(0,y) = v_0(y) \\ v_t(0,y) = v_1(y) \end{cases} & y \in]0,\infty[\,, \end{cases}$$

where $c(\theta) = \theta(3\theta-4)/4$,
$v_0(y) = \psi(y)^{\theta/4} u_0\big(\psi(y)\big),\, v_1(y) = \psi(y)^{\theta/4} u_1\big(\psi(y)\big)$

Given $T > 0$ and $\theta > 2$, consider the problem

$$\begin{cases} u_{tt} - \left(x^\theta u_x\right)_x = 0 &]0, T[\times]0,1[\\ \text{boundary conditions: } u(t,1) = 0 \text{ and } \lim_{x\downarrow 0} x^\theta u_x(t,x) = 0 & 0 < t < T \\ \text{initial conditions: } \begin{cases} u(0,x) = u_0(x) \\ u_t(0,x) = u_1(x) \end{cases} & x \in]0,1[, \end{cases}$$

where u_0 and u_1 are smooth functions with compact support in $]0,1[$

We now show that, for any fixed $T > 0$ the observability constant $C_T(\theta)$ of (1) for $a(x) = x^\theta$, with $0 \leqslant \theta < 2$, goes to zero as $\theta \uparrow 2$. We recall spectral re the sults (see Patrick Martinez' talk)

$$\begin{cases} -(x^\theta y'(x))' = \lambda y(x) & x \in]0,1[\\ \lim_{x\downarrow 0} x^\theta \, y'(x) = 0 \ \text{ and } \ y(1) = 0\,. \end{cases}$$

For any $\nu \geqslant 0$, denote by J_ν the Bessel function of the first kind of order ν, that is,

$$J_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\nu+1)} \left(\frac{x}{2}\right)^{2m+\nu} \qquad (x \geqslant 0),$$

where Γ is Euler's Gamma function.

Let j_ν be the first positive zero of J_ν.
Given $\theta \in [1, 2[$ define

$$\nu_\theta = \frac{\theta - 1}{2 - \theta} \quad \text{and} \quad \kappa_\theta = \frac{2 - \theta}{2}.$$

Then the first eigenvalue is given by $\lambda_\theta = \kappa_\theta^2 j_{\nu_\theta}^2$ and the corresponding normalized eigenfunction is

$$y_\theta(x) = \frac{\sqrt{2\kappa_\theta}}{\left|J'_{\nu_\theta}(j_{\nu_\theta})\right|} x^{\frac{1-\theta}{2}} J_{\nu_\theta}\left(j_{\nu_\theta} x^{\kappa_\theta}\right) \qquad (0 < x < 1).$$

For any fixed $T > 0$ the observability constant $C_T(\theta)$ of (1), with $1 \leqslant \theta < 2$, satisfies

$$C_T(\theta) \leqslant (2 - \theta) T.$$

Define

$$u_\theta(t,x) = \sin\left(\sqrt{\lambda_\theta}\, t\right) y_\theta(x) \qquad (t,x) \in]0,T[\times]0,1[.$$

Then u_θ satisfies $(1)_\theta$ with $u_0 \equiv 0$ and $u_1(x) = \sqrt{\lambda_\theta}\, y_\theta(x)$. Now, straightforward computations lead to

$$\frac{\int_0^T |\partial_x u_\theta|^2(t,1)\, dt}{E_{u_\theta}(0)} = 2T\kappa_\theta\left(1 - \frac{\sin\left(2\sqrt{\lambda_\theta}\, T\right)}{2\sqrt{\lambda_\theta}\, T}\right) < (2-\theta)T$$

taking into account the definition of κ_θ. The conclusion follows recalling the definition of C_T.

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Control in the coefficients of the p-Laplace problem

Joint work with E. Casas and P. Kogut (SICON 2016)

13 [illegible] 17

Let Ω be a bounded open subset of $\mathbb{R}^N$ ($N \geq 1$) with a Lipschitz boundary. Let p be a real number such that $2 \leq p < \infty$. By $BV(\Omega)$ we denote the space of all functions in $L^1(\Omega)$ for which the norm

$$\|f\|_{BV(\Omega)} = \|f\|_{L^1(\Omega)} + \int_\Omega |Df|$$

$$= \|f\|_{L^1(\Omega)} + \sup\left\{ \int_\Omega f \operatorname{div}\varphi \, dx \; : \; \varphi \in C_0^1(\Omega; \mathbb{R}^N), \; |\varphi(x)| \leq 1, x \in \Omega \right.$$

is finite.Let ξ_1, ξ_2 be given elements of $L^\infty(\Omega) \cap BV(\Omega)$ satisfying the conditions

$$0 < \alpha \leq \xi_1(x) \leq \xi_2(x) \text{ a.e. in } \Omega,$$

where α is a given positive value. Define the p-Laplacian

$$\boxed{\Delta_p(u, y) = \operatorname{div}\left(u(x)|\nabla y|^{p-2}\nabla y\right)}$$

where$|\nabla y|^{p-2} := |\nabla y|_{\mathbb{R}^N}^{p-2} = \left(\sum_{i=1}^N \left|\frac{\partial y}{\partial x_i}\right|^2\right)^{\frac{p-2}{2}}$, We consider the following optimal control problem

$$\text{Minimize } \left\{ I(u,y) = \int_\Omega |y - z_d|^2 \, dx + \int_\Omega |Du|, u \in \mathfrak{A}_{ad} \right\}$$

$$-\Delta_p(u,y) + y = f \quad \text{in} \quad \Omega,$$
$$y = 0 \quad \text{on} \quad \partial\Omega,$$

$$\mathfrak{A}_{ad} = \left\{ u \in BV(\Omega) \,\middle|\, \xi_1(x) \leq u(x) \leq \xi_2(x) \text{ a.e. in } \Omega \right\}.$$

Notice that for $p > 2$ the Problem exhibits degeneration on the set $\{x \,|\, \nabla y(x) = 0\}$. Moreover, degeneration may happen, where ξ_1 is zero.

As usual, a function $y \in W_0^{1,p}(\Omega)$ is said to be a solution

$$\boxed{\int_\Omega u|\nabla y|^{p-2}(\nabla y, \nabla\varphi)_{\mathbb{R}^N}\, dx + \int_\Omega y\varphi\, dx = \int_\Omega f\varphi\, dx \quad \forall\varphi \in W_0^{1,p}(\Omega).}$$

The existence of a unique solution to the boundary value problem follows from an abstract theorem on monotone operators.

Let V be a reflexive separable Banach space. Let V^* be the dual space, and let $A : V \to V^*$ be a bounded, semicontinuous, coercive and strictly monotone operator. Then the equation $Ay = f$ has a unique solution for each $f \in V^*$. Moreover, $Ay = f$ if and only if $\langle A\varphi, \varphi - y\rangle \geq \langle f, \varphi - y\rangle$ for all $\varphi \in V^*$.

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Let $\{(u_k, y_k)\}_{k\in\mathbb{N}} \subset \Xi$ be a bounded sequence. Then, there is a pair $(u, y) \in \Xi$ such that, up to a subsequence, $u_k \overset{*}{\rightharpoonup} u$ in $BV(\Omega)$ and $y_k \rightharpoonup y$ in $W_0^{1,p}(\Omega)$.

Proof: By compactness properties of $BV(\Omega) \times W_0^{1,p}(\Omega)$, there exists a subsequence of $\{(u_k, y_k)\}_{k\in\mathbb{N}}$ and functions $u \in BV(\Omega)$ and $y \in W_0^{1,p}(\Omega)$ such that $u_k \overset{*}{\rightharpoonup} u$ in $BV(\Omega)$, $y_k \rightharpoonup y$ in $W_0^{1,p}(\Omega)$. Then, we have

$$\lim_{k\to\infty} \int_\Omega (\nabla\varphi, \nabla y_k)_{\mathbb{R}^N}\, u_k\, dx = \int_\Omega (\nabla\varphi, \nabla y)_{\mathbb{R}^N}\, u\, dx, \quad \forall\, \varphi \in C_0^\infty(\Omega).$$

The Minty relation for (u_k, y_k) reads:

$$\int_\Omega |\nabla\varphi|^{p-2} (\nabla\varphi, \nabla\varphi - \nabla y_k)_{\mathbb{R}^N}\, u_k\, dx \geq \int_\Omega f(\varphi - y_k)\, dx, \quad \forall\, \varphi \in C_0^\infty(\Omega).$$

We can pass to the limit in relation as $k \to \infty$ and arrive at the Minity inequality for every $\varphi \in C_0^\infty(\Omega)$ and then for $W_0^{1,p}(\Omega)$, hence, $y \in W_0^{1,p}(\Omega)$ is solution to the boundary value problem.

Let $z_d \in L^2(\Omega)$ and $f \in L^2(\Omega)$ be given functions. Then optimal control problem admits at least one solution.

Proof: Since the set Ξ is nonempty and the cost functional is bounded from below on Ξ, it follows that there exists a minimizing sequence $\{(u_k, y_k)\}_{k\in\mathbb{N}} \subset \Xi$, i.e.

$$\inf_{(u,y)\in\Xi} I(u,y) = \lim_{k\to\infty} \left[\int_\Omega |y_k - z_d|^2\, dx + \int_\Omega |Du_k| \right] < +\infty.$$

Hence, $\{(u_k, y_k)\}_{k\in\mathbb{N}}$ is bounded in $BV(\Omega) \times W_0^{1,p}(\Omega)$. We deduce the existence of a subsequence, and a pair $(u^*, y^*) \in \Xi$ such that $u_k \overset{*}{\rightharpoonup} u$ in $BV(\Omega)$ and $y_k \rightharpoonup y^*$ in $W_0^{1,p}(\Omega)$. From these convergences we infer that

$$\lim_{k\to\infty} \int_\Omega |y_k - z_d|^2\, dx = \int_\Omega |y^* - z_d|^2\, dx \quad \text{and} \quad \liminf_{k\to\infty} \int_\Omega |Du_k| \geq \int_\Omega |Du^*|$$

So $I(u^*, y^*) \leq \inf_{(u,y)\in\Xi} I(u,y)$ and, consequently, (u^*, y^*) is a solution.

$$\text{Minimize } \left\{ I_{\varepsilon,k}(u,y) = \int_\Omega |y - z_d|^2\, dx + \int_\Omega |Du| \right\}$$

subject to the constraints

$$-\Delta_{\varepsilon,k,p}(u,y) + y = f \quad \text{in} \quad \Omega,$$
$$y = 0 \quad \text{on} \quad \partial\Omega,$$
$$u \in \mathfrak{A}_{ad} = \left\{ v \in BV(\Omega) \;\middle|\; \xi_1(x) \le v(x) \le \xi_2(x) \text{ a.e. in } \Omega \right\}.$$

$$\Delta_{\varepsilon,k,p}(u,y) := \operatorname{div}\left(u(x) \left(\varepsilon + \mathcal{F}_k\left(|\nabla y|^2\right)\right)^{\frac{p-2}{2}} \nabla y \right),$$

$\mathcal{F}_k : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-decreasing $C^1(\mathbb{R}_+)$-function such that

$$\mathcal{F}_k(t) = t, \quad \text{if} \quad t \in [0, k^2], \quad \mathcal{F}_k(t) = k^2 + 1, \quad \text{if} \quad t > k^2 + 1, \quad \text{and}$$
$$t \le \mathcal{F}_k(t) \le t + \delta, \quad \text{if} \quad k^2 \le t < k^2 + 1 \quad \text{for some} \quad \delta \in (0,1).$$

For each $\varepsilon > 0$, $k \in \mathbb{N}$, $u \in \mathfrak{A}_{ad}$, and $f \in L^2(\Omega)$, the regularized boundary value problem admits a unique weak solution $y_{\varepsilon,k} \in H_0^1(\Omega)$, i.e.

$$\int_\Omega u(\varepsilon + \mathcal{F}_k(|\nabla y_{\varepsilon,k}|^2))^{\frac{p-2}{2}} (\nabla y_{\varepsilon,k}, \nabla\varphi)_{\mathbb{R}^N}\, dx + \int_\Omega y_{\varepsilon,k}\varphi\, dx = \int_\Omega f\varphi\, dx,$$

$\forall\, \varphi \in H_0^1(\Omega)$, or equivalently

$$\int_\Omega u(x)(\varepsilon + \mathcal{F}_k(|\nabla\varphi|^2))^{\frac{p-2}{2}} (\nabla\varphi, \nabla\varphi - \nabla y_{\varepsilon,k})_{\mathbb{R}^N}\, dx$$
$$+ \int_\Omega \varphi(\varphi - y_{\varepsilon,k})\, dx \geq \int_\Omega f(\varphi - y_{\varepsilon,k})\, dx, \quad \forall\, \varphi \in C_0^\infty(\Omega).$$

For every positive value $\varepsilon > 0$ and integer $k \in \mathbb{N}$, the regularized optimal control problem has at least one solution.

Let $\left\{\left(u^0_{\varepsilon,k}, y^0_{\varepsilon,k}\right)\right\}_{\substack{\varepsilon>0\\ k\in\mathbb{N}}}$ be an arbitrary sequence of optimal pairs to the perturbed problems. Then, this sequence is bounded in $BV(\Omega)\times H^1_0(\Omega)$ and any cluster point (u^0, y^0) with respect to the (weak-*,weak) topology is a solution of the original OCP. Moreover, if for one subsequence we have $u^0_{\varepsilon,k} \overset{*}{\rightharpoonup} u^0$ in $BV(\Omega)$ and $y^0_{\varepsilon,k} \rightharpoonup y^0$ in $H^1_0(\Omega)$, then the following properties hold

$$\lim_{\substack{\varepsilon\to0\\ k\to\infty}} (u^0_{\varepsilon,k}, y^0_{\varepsilon,k}) = (u^0, y^0) \quad \text{strongly in} \quad L^1(\Omega)\times H^1_0(\Omega),$$

$$\lim_{\substack{\varepsilon\to0\\ k\to\infty}} \int_\Omega |u^0_{\varepsilon,k}| = \int_\Omega |Du^0|,$$

$$\lim_{\substack{\varepsilon\to0\\ k\to\infty}} \chi_{\Omega_k(y^0_{\varepsilon,k})}\nabla y^0_{\varepsilon,k} = \nabla y^0 \quad \text{strongly in} \quad L^p(\Omega)^N,$$

$$\lim_{\substack{\varepsilon\to0\\ k\to\infty}} \int_\Omega \left(\varepsilon + \mathcal{F}_k(|\nabla y^0_{\varepsilon,k}|^2)\right)^{\frac{p-2}{2}} |\nabla y^0_{\varepsilon,k}|^2 u^0_{\varepsilon,k}\, dx = \int_\Omega |\nabla y^0|^p u^0\, dx,$$

$$\lim_{\substack{\varepsilon\to0\\ k\to\infty}} I_{\varepsilon,k}(u^0_{\varepsilon,k}, y^0_{\varepsilon,k}) = I(u^0, y^0).$$

Damage modeling and sustainable optimal control

Joint work with P. Kogut (Math. Meth. Appl. Sci. 2013)

$$\exists\, \zeta_*^0 \in \Psi_* \text{ such that } \zeta_*^0 \leq \zeta_0 \leq 1 \quad \text{a.e. in} \quad \Omega,$$

a displacement field $\mathbf{u} : \Omega_T \to \mathbb{R}^N$, a stress field $\boldsymbol{\sigma} : \Omega_T \to \mathbb{S}^N$, and a damage field $\zeta : \Omega_T \to \mathbb{R}$ satisfy the relations

$$-\operatorname{div} \boldsymbol{\sigma} = \mathbf{f} \quad \text{in} \quad \Omega_T,$$

$$\boldsymbol{\sigma} = \zeta A \mathbf{e}(\mathbf{u}) \quad \text{in} \quad \Omega_T,$$

$$\mathbf{u} = 0 \quad \text{on} \quad (0,T) \times S,$$

$$\boldsymbol{\sigma\nu} = \mathbf{p} \quad \text{on} \quad (0,T) \times \Gamma, \ \mathbf{p} \in \mathcal{P}_{ad},$$

$$\zeta' - \kappa \Delta \zeta = \phi(\mathbf{e}(\mathbf{u}), \zeta) \quad \text{in} \quad \Omega_T,$$

$$\zeta(0,\cdot) = \zeta_0 \quad \text{in} \quad \Omega,$$

$$\zeta = 1 \quad \text{on } (0,T) \times \Gamma, \quad \frac{\partial \zeta}{\partial n} = 0 \quad \text{on } (0,T) \times S,$$

$$\exists\, \zeta_* \in \Psi_* \text{ such that } \zeta_* \leq \zeta(t,x) \leq 1 \quad \text{a.e. in} \quad \Omega_T.$$

Happy birthday Piermarco!

sarà tre volte Natale
e festa tutto il giorno!

FRIEDRICH-ALEXANDER UNIVERSITÄT ERLANGEN-NÜRNBERG

Thank you for your attention and come visit us at FAU!