Kuramoto-Lohe synchronization over quantum networks

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(joint work with P. Antonelli)

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Applications Examples

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Introduction and Classical K.

Schrödinger – Lohe

Order Parameter Correlations

N=2

Macro Correlations

L^2 and H^1 Synchro

Wigner N=2

Quantum Hydrodynamics

Conclusion

- Classical: Huygens' observation (1665) two pendulum clocks fastened to the same beam will synchronize (anti-phase)
- Classical:rhythmic applause in a large audience
- Classical: synchronous flashing of fireflies
- Biology: Classical Winfree and Kuramoto
- Quantum: Van der Pol oscillators
- Quantum Synchronization in microsystems
- Quantum Cryptography

FROM ECKHAUS to LANDAU-STUART

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Consider the ECKHAUS equation

$$i\psi_t + \psi_{xx} + 2\left(|\psi|^2\right)_x \psi + |\psi|^4 \psi = 0$$

Transformed into $i\varphi_t + \varphi_{xx} = 0$, via the Calogero – De Lillo transformation (complete integrability)

$$\psi(x,t) = \frac{\varphi(x,t)}{\left(1 + 2\int_{-\infty}^{x} |\varphi(x',t)|^2 \, \mathrm{d}x'\right)^{1/2}}.$$

Asymptotics of the ECKHAUS EQUATION regarding looking for a small-amplitude equation valid near the Hopf bifurcation point lead to the LANDAU-STUART equations

$$\partial_t A = \chi A - g|A|^2 A$$

Actually emerges quite generically in systems close to bifurcation (Weakly Nonlinear Dynamics) . See Kuramoto's book

Classical Kuramoto model

Complete network of N-nodes with edges connecting all pair of nodes. Let $z_i \in \mathbb{C}^1$ the state of the i-th Landau-Stuart oscillator at each node.
Then z_i governed by

$$\frac{dz_i}{dt} = (1 - |z_i|^2 + \mathrm{i}\omega_i)z_i + \frac{K}{N}\sum_{j=1}^{N}(z_j - z_i), \quad j = 1, \cdots, N, \quad (1)$$

K is the uniform coupling strength between oscillators, ω_i is the quenched random natural frequency of the i-th Stuart-Landau oscillator extracted from a given distribution function $g = g(\omega)$, $\omega \in \mathbb{R}$, supp $g(\cdot) \subset\subset \mathbb{R}$:

$$\int_{\mathbb{R}} g(\omega)\, d\omega = 1, \quad \int_{\mathbb{R}} \omega g(\omega)\, d\omega = 0, \quad g(\omega) \geq 0.$$

From Landau–Stuart to Kuramoto following Kuramoto

The state $z_i = z_i(t)$ governed by (1) approaches a limit-cycle (a circle with radius determined by the coupling strength) asymptotically for a suitable range of K. Ignore the amplitude variations and focus on the phase dynamics ("*weakly coupled oscillator*"). Let

$$z_i(t) := e^{\mathrm{i}\theta_i(t)}, \quad t \geq 0, \quad 1 \leq i \leq N, \tag{2}$$

plug (2) into (1), use the imaginary part of the resulting relation

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i), \quad i = 1, \cdots, N. \tag{3}$$

which is the so called Kuramoto model.

PHASE LOCKING and SYNCHRONIZATION

Definition

Let $\Theta = (\theta_1, \cdots, \theta_N)$ be a solution to (3).

1. Θ is a "*Phase-locked state*" if

$$|\theta_i(t) - \theta_j(t)| = |\theta_i(0) - \theta_j(0)|, \quad \forall\, t \geq 0, \quad 1 \leq i, j \leq N.$$

2. "*complete (frequency) synchronization*" asymptotically if:

$$\lim_{t\to\infty} \max_{1\leq i,j\leq N} |\dot{\theta}_i(t) - \dot{\theta}_j(t)| = 0.$$

3. "*(practical) complete phase synchronization*" asymptotically if:

$$\lim_{K\to\infty} \lim_{t\to\infty} \max_{1\leq i,j\leq N} |\theta_i(t) - \theta_j(t)| = 0.$$

The Schrödinger – Lohe system – Quantum Oscillators

Consider wave functions ψ_j, with $\psi_j : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}, \quad j = 1, \dots, N$, where the dynamics are given by

$$i\partial_t \psi_j = -\frac{1}{2}\Delta\psi_j + V_j\psi_j + i\frac{K}{2N}\sum_{\ell=1}^{N}\left(\psi_\ell - \frac{\langle \psi_\ell, \psi_j\rangle}{\|\psi_j\|_{L^2}^2}\psi_j\right) \tag{4}$$

The potentials V_j are given by $V_j = V + \Omega_j$, where V is an external potential and the $\Omega_j \in \mathbb{R}$ are given constants (in real models random variables), the natural frequencies for the classical Kuramoto model.

Standard Existence and Uniqueness

By using the standard literature (e.g. Cazenave or T.Tao books) we have

Proposition (Existence and Uniqueness)

Let $\psi_{j,0} \in L^2(\mathbb{R}^d)$ for any $j = 1, \dots, N$, then there exists a unique global solution $(\psi_1, \dots, \psi_N) \in \mathcal{C}(\mathbb{R}_+; L^2(\mathbb{R}^d))$ to the system (4), with initial data $\psi_j(0) = \psi_{j,0}$. Furthermore the total mass of each individual wave function is conserved, i.e.

$$\|\psi_j(t)\|_{L^2} = \|\psi_{j,0}\|_{L^2}, j = 1, \dots, N.$$

for all t>0.

Moreover, let $\psi_{j,0} \in H^1(\mathbb{R}^d)$, then $\psi_j \in \mathcal{C}(\mathbb{R}_+; H^1(\mathbb{R}^d))$.

$$\frac{d}{dt}\left(\frac{1}{2}\int |\psi_j(t,x)|^2\,dx\right)$$
$$= \frac{K}{2N}\sum_{\ell=1}^{N}\mathrm{Re}\left(\langle\psi_j,\psi_\ell\rangle - \frac{\langle\psi_\ell,\psi_j\rangle}{\|\psi_j\|_{L^2}^2}\|\psi_j\|_{L^2}^2\right) = 0.$$

Assume $\|\psi_j(0)\|_{L^2} = 1$, $j = 1, \ldots, N$. Hence

$$i\partial_t \psi_j = -\frac{1}{2}\Delta\psi_j + V_j\psi_j + i\frac{K}{2N}\sum_{\ell=1}^{N}\left(\psi_\ell - \langle\psi_\ell, \psi_j\rangle\psi_j\right).$$

Assume $\sum_j \Omega_j = 0$ otherwise if $\frac{1}{N}\sum_j \Omega_j = \alpha \neq 0$, then

$$\psi_j'(t,x) = e^{-i\alpha t}\psi_j(t,x), \qquad \forall\, j = 1, \ldots, N.$$

Order Parameter

As in Kuramoto the **order parameter** $\zeta := \frac{1}{N}\sum_{\ell=1}^{N}\psi_\ell$.

$$i\partial_t\zeta = -\frac{1}{2}\Delta\zeta + V\zeta + \frac{1}{N}\sum_{\ell=1}^{N}\Omega_\ell\psi_\ell + i\frac{K}{2}\left(\zeta - \frac{1}{N}\sum_{\ell=1}^{N}\langle\zeta,\psi_\ell\rangle\psi_\ell\right).$$

Definition

Complete frequency synchronization $\quad j,k = 1,\dots,N$

$$\lim_{t\to\infty}\|\psi_j(t)-\psi_k(t)\|_{L^2} = c_{jk},\ \lim_{t\to\infty}\|\zeta(t)\|_{L^2} = c \in (0,1).$$

Complete phase synchronization

$$\lim_{t\to\infty}\|\psi_j(t)-\psi_k(t)\|_{L^2} = 0,\ \lim_{t\to\infty}\|\zeta(t)\|_{L^2} = 1.$$

Uncorrelated $1 - \|\zeta\|_{L^2}^2 = \frac{1}{2N^2}\sum_{j,k=1}^{N}\|\psi_j-\psi_k\|_{L^2}^2$

Quantum Correlations

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Correlations among wave functions

$$r_{jk} := \mathrm{Re}\langle \psi_j, \psi_k \rangle, \quad s_{jk} := \mathrm{Im}\langle \psi_j, \psi_k \rangle \quad r_{jk} = r_{kj}, \; s_{jk} = -s_{kj}.$$

The system of ODEs describing the coupled dynamics

Proposition (Decoupled ODE system for Correlations)

$$\frac{d}{dt} r_{jk} = -(\Omega_j - \Omega_k) s_{jk} + \frac{K}{2N} \sum_{\ell=1}^{N} \left[(r_{j\ell} + r_{\ell k})(1 - r_{jk}) + (s_{j\ell} + s_{\ell k}) s_{jk} \right]$$

$$\frac{d}{dt} s_{jk} = (\Omega_j - \Omega_k) r_{jk} + \frac{K}{2N} \sum_{\ell=1}^{N} \left[-(r_{j\ell} + r_{\ell k}) s_{jk} + (s_{j\ell} + s_{\ell k})(1 - r_{jk}) \right].$$

Correlations w.r.t. Order Parameter (Macroscopic Correlations)

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$$\tilde{r}_j := \mathrm{Re}\langle \zeta, \psi_j \rangle = \frac{1}{N}\sum_{\ell=1}^N r_{\ell j}, \quad \tilde{s}_j := \mathrm{Im}\langle \zeta, \psi_j \rangle = \frac{1}{N}\sum_{\ell=1}^N s_{\ell j}.$$

$$\frac{1}{N}\sum_{j=1}^N \tilde{r}_j = \|\zeta\|_{L^2}^2, \qquad \frac{1}{N}\sum_{j=1}^N \tilde{s}_j = 0.$$

$$\frac{d}{dt}\|\zeta(t)\|_{L^2}^2 = \frac{2}{N}\sum_{\ell=1}^N \Omega_\ell \tilde{s}_\ell + K\left(\|\zeta(t)\|_{L^2}^2 - \frac{1}{N}\sum_{\ell=1}^N (\tilde{r}_\ell^2 - \tilde{s}_\ell^2)\right).$$

Proposition

The macroscopic correlations satisfy

$$\begin{cases} \frac{d}{dt}\tilde{r}_j = \Omega_j \tilde{s}_j - \frac{1}{N}\sum_{\ell=1}^N \Omega_\ell s_{\ell j} + \frac{K}{2}\left[\tilde{r}_j - \tilde{r}_j^2 + \tilde{s}_j^2 + \frac{1}{N}\sum_{\ell=1}^N (\tilde{r}_\ell - \tilde{r}_\ell r_{\ell j} - \tilde{s}_\ell s_{\ell j})\right] \\ \frac{d}{dt}\tilde{s}_j = -\Omega_j \tilde{r}_j + \frac{1}{N}\sum_{\ell=1}^N \Omega_\ell r_{\ell j} + \frac{K}{2}\left[\tilde{s}_j - 2\tilde{r}_j\tilde{s}_j + \frac{1}{N}\sum_{\ell=1}^N (r_{\ell j}\tilde{s}_\ell - s_{\ell j}\tilde{r}_\ell)\right]. \end{cases}$$

Correlations dynamics for N=2

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Theorem

Let $\Lambda := \frac{2\Omega}{K} \geq 0$, $\psi_{1,0}, \psi_{2,0} \in L^2(\mathbb{R}^d)$, *then*

1. *for* $0 \leq \Lambda < 1$, $\langle \psi_{1,0}, \psi_{2,0} \rangle \neq -\sqrt{1-\Lambda^2} + i\Lambda$, *complete synchronization,*

$$\|e^{i\phi}\psi_1(t) - \psi_2(t)\|_{L^2} \lesssim e^{-\sqrt{K^2-4\Omega^2}t}, \quad as\ t \to \infty$$

$$\lim_{t\to\infty} \|\psi_1(t) - \psi_2(t)\|_{L^2} = \left|1 - e^{i\phi}\right|, \sin^{-1}\Lambda = \phi$$

2. *for* $\Lambda = 1$, $\langle \psi_{1,0}, \psi_{2,0} \rangle \neq i$,

$$\|i\psi_1(t) - \psi_2(t)\|_{L^2} \lesssim t^{-1}, \quad as\ t \to \infty$$

$$\lim_{t\to\infty} \|\psi_1(t) - \psi_2(t)\|_{L^2} = \sqrt{2};$$

3. *for* $\Lambda > 1$, *the correlations are periodic in time.*

Correlation ODE N=2

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$$z(t) := \langle \psi_1, \psi_2 \rangle (t)$$

$\Lambda < 1$ $\quad z_1 = \sqrt{1-\Lambda^2} + i\Lambda =: e^{i\phi}$, $z_2 = -\sqrt{1-\Lambda^2} + i\Lambda$

$$\dot{z} = 2i\Omega z + \frac{K}{2}(1-z^2), \quad z(0) \neq z_{1,2}$$

$$z(t) = \frac{e^{i\phi} + e^{-i\phi}\frac{z(0)-e^{i\phi}}{z(0)+e^{-i\phi}} e^{-\sqrt{K^2-4\Omega^2}t}}{1 - \frac{z(0)-e^{i\phi}}{z(0)+e^{i\phi}} e^{-\sqrt{K^2-4\Omega^2}t}},$$

$|z(t) - e^{i\phi}| \lesssim e^{-\sqrt{K^2-4\Omega^2}t}$ $\quad \|\psi_1\|_{L^2} = \|\psi_2\|_{L^2} = 1$

$$\lim_{t\to\infty} \|\psi_1(t) - \psi_2(t)\|_{L^2} = |1 - e^{i\phi}|.$$

$\Lambda = 1$, then $z_1 = z_2 = i$ then $z(t) = i + \left(\frac{K}{2}t + \frac{1}{z(0)-i}\right)^{-1}$.

Connection to Classical Kuramoto system

Remark

Let us consider again the previouus ODE, write formally $z = re^{i\theta}$, then (r, θ) solve

$$\begin{cases} \dot{r} = \frac{K}{2}\cos\theta(1 - r^2) \\ \dot{\theta} = 2\Omega - K\sin\theta + \frac{K}{2}(1 - r^2)\sin\theta, \end{cases}$$

$r \equiv 1$, $z(t) = e^{i\theta(t)}$, with $\theta(t)$ is a solution of the classical Kuramoto model, are also solutions to the previous ODE.

Phase plane analysis

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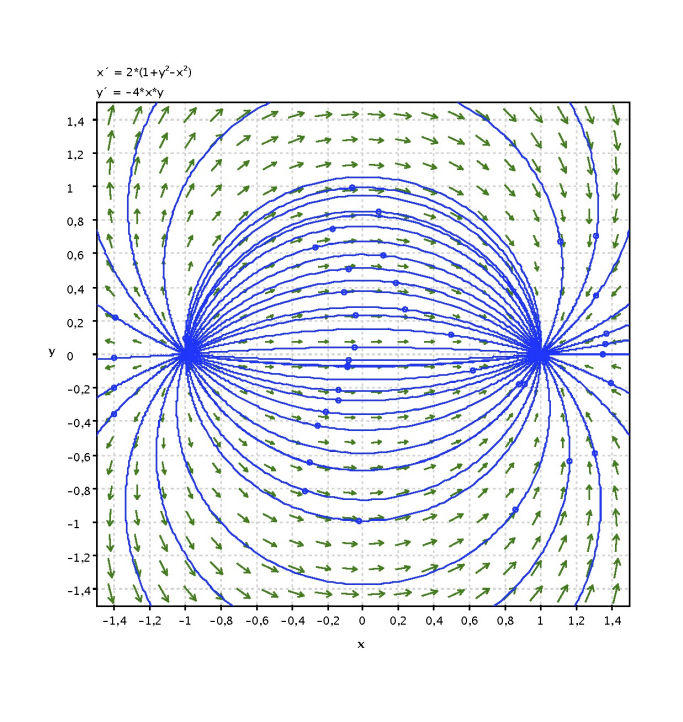
$$z(t) = r_{12}(t) + is_{12}(t),$$

$$\begin{cases} \dfrac{d}{dt} r_{12} = -2\Omega s_{12} + \dfrac{K}{2}(1 - r_{12}^2 + s_{12}^2) \\ \dfrac{d}{dt} s_{12} = 2\Omega r_{12} - K r_{12} s_{12}, \end{cases}$$



The case $N \geq 3$ Macro - Correlations

In the general case a key-role is played by **Macro Correlations**

$$\tilde{r}_j := \mathrm{Re}\langle \zeta, \psi_j \rangle = \frac{1}{N}\sum_{\ell=1}^N r_{\ell j}, \qquad \tilde{s}_j := \mathrm{Im}\langle \zeta, \psi_j \rangle = \frac{1}{N}\sum_{\ell=1}^N s_{\ell j}.$$

$$\frac{d}{dt}\|\zeta(t)\|_{L^2}^2 = \frac{2}{N}\sum_{\ell=1}^N \Omega_\ell \tilde{s}_\ell + K\left(\|\zeta(t)\|_{L^2}^2 - \frac{1}{N}\sum_{\ell=1}^N(\tilde{r}_\ell^2 - \tilde{s}_\ell^2)\right).$$

$$\begin{aligned}\frac{d}{dt}\tilde{r}_j &= \Omega_j \tilde{s}_j - \frac{1}{N}\sum_{\ell=1}^N \Omega_\ell s_{\ell j} \\ &\quad + \frac{K}{2}\left[\tilde{r}_j - \tilde{r}_j^2 + \tilde{s}_j^2 + \frac{1}{N}\sum_{\ell=1}^N(\tilde{r}_\ell - \tilde{r}_\ell r_{\ell j} - \tilde{s}_\ell s_{\ell j})\right]\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}\tilde{s}_j &= -\Omega_j \tilde{r}_j + \frac{1}{N}\sum_{\ell=1}^N \Omega_\ell r_{\ell j} \\ &\quad + \frac{K}{2}\left[\tilde{s}_j - 2\tilde{r}_j\tilde{s}_j + \frac{1}{N}\sum_{\ell=1}^N(r_{\ell j}\tilde{s}_\ell - s_{\ell j}\tilde{r}_\ell)\right].\end{aligned}$$

Asymptotic States

Assume $\Omega_j \equiv 0$ for any $j = 1, \ldots, N$.

Proposition

We have

$$\zeta - \langle \zeta, \psi_j \rangle \psi_j = 0, \qquad \forall\, j = 1, \ldots, N,$$

if and only if one of the two cases hold:

- $\zeta = 0$ *(incoherent dynamics);*
- *upon relabelling the wave functions, we have* $\psi_1 = \ldots = \psi_k = -\psi_{k+1} = \ldots = -\psi_N$, *for some* $k > \frac{N}{2}$ *and consequently*

$$\zeta = \left(\frac{2k}{N} - 1 \right) \psi_1.$$

Decay macroscopic Correlations

In this context, we investigate when the system exhibits complete phase synchronization.

Proposition

Let $(\psi_1, \dots, \psi_N)$ be solutions to Schrödinger-Lohe, with $(\psi_1(0), \dots, \psi_N(0)) = (\psi_{1,0}, \dots, \psi_{N,0})$ and with $\Omega_j = 0$ for any $j = 1, \dots, N$. Let us furthermore assume $\tilde{r}_j(0) > 0$ for any $j = 1, \dots, N$. Then we have

$$|1 - \tilde{r}_j(t)|^2 + |\tilde{s}_j(t)|^2 \lesssim e^{-Kt}, \quad as\ t \to \infty.$$

Namely $|1 - \langle \zeta, \psi_j(t) \rangle| \lesssim e^{-Kt}$, as $t \to \infty$

$$1 - \langle \zeta, \psi_j \rangle =: f_j + i g_j$$

$$\frac{d}{dt} f_j = -\frac{K}{2}\left[2f_j - (f_j^2 - g_j^2) - \frac{1}{N}\sum_{\ell=1}^{N}(f_\ell f_{\ell j} + g_\ell g_{\ell j})\right]$$

$$\frac{d}{dt} g_j = -\frac{K}{2}\left[2g_j - 2f_j g_j - \frac{1}{N}\sum_{\ell=1}^{N}(f_\ell g_{\ell j} - g_\ell f_{\ell j})\right].$$

$$\frac{d}{dt}\left(\frac{1}{2N}\sum_{j=1}^{n}(f_j^2 + g_j^2)\right)$$

$$= -\frac{K}{2N}\sum_{j=1}^{N}(2 - f_j)(f_j^2 + g_j^2) + \frac{K}{2N^2}\sum_{j,\ell=1}^{N} f_{\ell j}(f_j f_\ell + g_j g_\ell)$$

$$\frac{1}{2N}\sum_{j=1}^{N}\left(f_j^2 + g_j^2\right)(t) \leq e^{-Kt}\frac{1}{2N}\sum_{j=1}^{N}(f_j(0)^2 + g_j(0)^2)$$

L^2 Synchronization

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Proposition

We have

$$\lim_{t\to\infty} \|\psi_j(t) - \psi_k(t)\|_{L^2} = 0, \qquad \forall\, j,k = 1,\dots,N$$

and moreover

$$\|\psi_j(t) - \psi_k(t)\|_{L^2} \lesssim e^{-Kt}, \quad as\ t \to \infty.$$

H^1 Synchronization

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Theorem

Let $(\psi_1, \dots, \psi_N)$ be solutions to S-L, with $\psi_1(0), \dots, \psi_N(0)) = (\psi_{1,0}, \dots, \psi_{N,0}) \in H^1(\mathbb{R}^d)$ and $\Omega_j = 0$ for any $j = 1, \dots, N$. Let $\tilde{r}_j(0) > 0$. Then we have

$$\lim_{t\to\infty} \|\psi_j(t) - \psi_k(t)\|_{H^1} = 0, \quad as\, j, k = 1, \dots, N.$$

$$E(t) = \frac{1}{N}\sum_{j=1}^{N} E_j(t), \quad E_j(t) = \int \frac{1}{2}|\nabla\psi_j|^2 + V|\psi_j|^2\, dx$$

$$\frac{d}{dt}E(t) = \frac{1}{N}\sum_{j=1}^{N} 2\int \mathrm{Re}\left\{\left(-\frac{1}{2}\Delta\bar{\psi}_j + V\bar{\psi}_j\right)\frac{K}{2}(\zeta - \langle\zeta, \psi_j\rangle\psi_j)\right\} dx$$

$$= -\frac{K}{N}\sum_{j=1}^{N}\tilde{r}_j(t)E_j(t) + \frac{K}{N}\sum_{j=1}^{N}\int \mathrm{Re}\left\{\frac{1}{2}\nabla\bar{\psi}_j\cdot\nabla\zeta + V\bar{\psi}_j\zeta\right\} dx.$$

Relative Energy

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$$\mathbf{E_{jk}(t)} = \int \frac{\mathbf{1}}{\mathbf{2}}|\nabla(\psi_\mathbf{j} - \psi_\mathbf{k})|^\mathbf{2} + \mathbf{V}|\psi_\mathbf{j} - \psi_\mathbf{k}|^\mathbf{2}\,\mathbf{dx}.$$

$$\begin{aligned}\frac{d}{dt}E(t) =& \frac{K}{N}\sum_{j=1}^{N}(1-\tilde{r}_j(t))E_j(t) - \frac{K}{2N^2}\sum_{j,k=1}^{N}E_{jk}(t)\\ \leq& \frac{K}{N}\sum_{j=1}^{N}(1-\tilde{r}_j(t))E_j(t).\end{aligned}$$

$$\tilde{\mathbf{E}}\mathbf{(t)} := \frac{\mathbf{1}}{\mathbf{2N^2}}\sum_{\mathbf{j,k=1}}^{\mathbf{N}}\mathbf{E_{jk}(t)} \quad \mathbf{E_z(t)} := \int \frac{\mathbf{1}}{\mathbf{2}}|\nabla\zeta|^\mathbf{2} + \mathbf{V}|\zeta|^\mathbf{2}\,\mathbf{dx}$$

$$\begin{aligned}E_z(t) =& \frac{1}{N^2}\sum_{j,k=1}^{N}\int \mathrm{Re}\left\{\frac{1}{2}\nabla\bar{\psi}_j\cdot\nabla\psi_k + V\bar{\psi}_j\psi_k\right\}dx\\ =& -\frac{1}{2N^2}\sum_{j,k=1}^{N}E_{jk}(t) + \frac{1}{N}\sum_{j=1}^{N}E_j(t),\end{aligned}$$

$$
\begin{aligned}
\frac{d}{dt}\tilde{E}(t) =& \frac{d}{dt}\left(E(t) - E_z(t)\right) \\
=& \frac{K}{N}\sum_{j=1}^{N}(1-\tilde{r}_j(t))E_j(t) - K\tilde{E}(t) \\
& - 2\int \mathrm{Re}\left\{\left(-\frac{1}{2}\Delta\bar{\zeta} + V\bar{\zeta}\right)\frac{K}{2}\left(\zeta - \frac{1}{N}\sum_{\ell=1}^{N}\langle\zeta,\psi_\ell\rangle\psi_\ell\right)\right\}dx \\
=& \frac{K}{N}\sum_{j=1}^{N}(1-\tilde{r}_j(t))E_j(t) - K\tilde{E}(t) - KE_z(t) \\
& + \frac{K}{N}\sum_{\ell=1}^{N}\mathrm{Re}\left\{\langle\zeta,\psi_\ell\rangle\int\frac{1}{2}\nabla\bar{\zeta}\cdot\nabla\psi_\ell + V\bar{\zeta}\psi_\ell\,dx\right\}.
\end{aligned}
$$

$$\frac{d}{dt}\tilde{E}(t) = -K\tilde{E}(t) + \frac{K}{2N^2}\sum_{j,k=1}^{N}(1-r_{jk}(t))E_{jk}(t)$$

$$-\frac{K}{N^2}\sum_{j,k=1}^{N} s_{jk}(t)\int \mathrm{Im}\left\{\frac{1}{2}\nabla\bar{\psi}_j\cdot\nabla\psi_k + V\bar{\psi}_j\psi_k\right\}dx$$

$$\leq -K\tilde{E}(t) + \frac{K}{2N^2}\sum_{j,k=1}^{N}(1-r_{jk}(t))E_{jk}(t) + \frac{K}{2N^2}\sum_{j,k=1}^{N}|s_{jk}(t)|E_j(t).$$

$$\tilde{E}(t) \leq e^{-Kt}\tilde{E}(0) + \frac{K}{2N^2}\sum_{j,k=1}^{N}\int_0^t e^{-K(t-s)}e^{-Ks}\left(E_{jk}(s)+E_j(s)\right)ds.$$

Wigner Equations

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This part is in collaboration also with SEUNG-YEAL HA and DOHYUN KIM

Definition (Wigner transform)

For any two wave functions $\psi, \phi \in L^2$, we define the Wigner transform

$$w[\psi,\phi](x,p) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{iy\cdot p}\bar{\psi}\left(x+\frac{y}{2}\right)\phi\left(x-\frac{y}{2}\right)\,dy.$$

$$\Theta[V](w)(x,p) := -\frac{i}{(2\pi)^d}\int e^{i(p-p')\cdot y}\left(V\left(x+\frac{y}{2}\right)-V\left(x-\frac{y}{2}\right)\right)w(x,p')\,dp'dy.$$

$$\partial_t w + p\cdot\nabla_x w + \Theta[V]w = 0,$$

where

$$i\partial_t\psi = -\frac{1}{2}\Delta\psi + V\psi,$$

Wigner Lohe System

$$
\begin{cases}
\partial_t w_j + p \cdot \nabla_x w_j + \Theta[V](w_j) = \dfrac{K}{N} \displaystyle\sum_{k=1}^{N} \Big[w_{jk}^+ - \Big(\int w_{jk}^+ dpdx \Big) w_j \Big], \\
\partial_t w_{jk}^+ + p \cdot \nabla_x w_{jk}^+ + \Theta[V](w_{jk}^+) \\
\qquad = \dfrac{K}{2N} \displaystyle\sum_{\ell=1}^{N} \Big[w_{j\ell}^+ + w_{\ell k}^+ - \Big(\int (w_{j\ell}^+ + w_{\ell k}^+) dpdx \Big) w_{jk}^+ + \Big(\int (w_{j\ell}^- + w_{\ell k}^-) dpdx \Big) \\
\partial_t w_{jk}^- + p \cdot \nabla_x w_{jk}^- + \Theta[V](w_{jk}^-) \\
\qquad = \dfrac{K}{2N} \displaystyle\sum_{\ell=1}^{N} \Big[w_{j\ell}^- + w_{\ell k}^- - \Big(\int (w_{j\ell}^+ + w_{\ell k}^+) dpdx \Big) w_{jk}^- + \Big(\int (w_{j\ell}^- + w_{\ell k}^-) dpdx \Big)
\end{cases}
$$

Wigner Synchronization N=2

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Introduction and Classical K.

Schrödinger – Lohe

Order Parameter Correlations

N=2

Macro Correlations

L^2 and H^1 Synchro

Wigner N=2

Quantum Hydrodynamics

Conclusion

Theorem

Let (w_1, w_2, w_{12}) *be a solution to W-L with initial data* $(w_1(0), w_2(0), w_{12}(0)) = (w_1^0, w_2^0, w_{12}^0)$ *s.t.*

$$\int w_1^0(x,p)\,dxdp = \int w_2^0(x,p)\,dxdp = 1,$$

and

$$\Big|\int w_{12}^0(x,p)\,dxdp\Big| \leq 1, \quad \int w_{12}^0(x,p)\,dxdp \neq -1.$$

Then we have

$$\|w_1(t) - w_2(t)\|_{L^2}^2 \leq e^{-Kt}, \quad as\ t \to \infty.$$

Quantum Hydrodynamics N=2

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Hydrodynamical quantities

$$\rho_a = \frac{1}{2}|\psi_a|^2, \;\; J_a = \frac{1}{2}\,\mathrm{Im}(\bar{\psi}_a \nabla \psi_a) = \sqrt{\rho_a}\Lambda_a$$

.

$$\begin{cases} \partial_t \rho_1 + \mathrm{div}\, J_1 = \dfrac{K}{2}(\rho_{12} - r_{12}\rho_1), \\ \partial_t \rho_2 + \mathrm{div}\, J_2 = \dfrac{K}{2}(\rho_{12} - r_{12}\rho_2), \\ \partial_t J_1 + \mathrm{div}\left(\dfrac{J_1 \otimes J_1}{\rho_1}\right) + \rho_1 \nabla V = \dfrac{1}{2}\rho_1 \nabla\left(\dfrac{\Delta\sqrt{\rho_1}}{\sqrt{\rho_1}}\right) + \dfrac{K}{2}(J_{12} - r_{12}J_1), \\ \partial_t J_2 + \mathrm{div}\left(\dfrac{J_2 \otimes J_2}{\rho_2}\right) + \rho_2 \nabla V = \dfrac{1}{2}\rho_2 \nabla\left(\dfrac{\Delta\sqrt{\rho_2}}{\sqrt{\rho_2}}\right) + \dfrac{K}{2}(J_{12} - r_{12}J_2). \end{cases}$$

$$\lim_{t\to\infty} \left(\|\nabla\sqrt{\rho_1}(t) - \nabla\sqrt{\rho_2}(t)\|_{L^2} + \|\Lambda_1(t) - \Lambda_2(t)\|_{L^2}\right) = 0.$$

Happy Birthday
Piermarco
Happy 60 !!!!
Felici 60