Necessary optimality conditions for infinite dimensional state constrained control problems

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Outline of the talk

- H. Frankowska, E.M. Marchini, M. M.
  Distance estimates for state constrained trajectories of infinite dimensional differential inclusions
  ESAIM Control Optim. Calc. Var., accepted

- H. Frankowska, E.M. Marchini, M. M.
  Necessary optimality conditions for infinite dimensional state constrained control problems
  in preparation

Semilinear Differential Inclusions

We consider the Differential Inclusion

$$\text{(DI)} \qquad x'(t) \in \mathbb{A}x(t) + F(t, x(t)) \quad \text{a.e. } t \in [t_0, 1], \quad x(t_0) = x_0$$

under the State Constraint

$$\text{(SC)} \qquad x(t) \in K \quad \text{for } t \in [t_0, 1].$$

- X an infinite dimensional separable Banach space;
- $\mathbb{A}$ is the infinitesimal generator of a strongly continuous semigroup $S(t) : X \to X$.
- $F : I \times X \rightsquigarrow X$ is a set-valued map with closed non-empty images, $I = [0, 1]$ and $t_0 \in I$;
- K is a nonempty closed subset of X.

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Example: Control system

$$\begin{cases} x'(t) = \mathbb{A}x(t) + f(t, x(t), u(t)) & \text{a.e. } t \in [t_0, 1] \\ u(t) \in U. \end{cases}$$

In this case, we set $F(t, x) = f(t, x, U)$.

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A function $x \in \mathcal{C}([t_0, 1], X)$ is a mild solution of (DI) with initial datum $x(t_0) = x_0$ if there exists a function $f_x \in L^1(t_0, 1; X)$ such that

$$f_x(t) \in F(t, x(t)) \qquad \text{for a.e. } t \in (t_0, 1)$$

and

$$x(t) = S(t - t_0)\, x_0 + \int_{t_0}^{t} S(t-s)\, f_x(s)\, \mathrm{d}s \qquad \text{for any } t \in [t_0, 1],$$

Assumptions (A)

(A1) $\forall\, x \in X$ the set-valued map $F(\cdot, x)$ is Lebesgue measurable;

(A2) $\exists\, k \in L^1(I; \boldsymbol{R}^+)$ such that, $F(t, \cdot)$ is $k(t)$-Lipschitz for a.e. $t \in I$, i.e.

$$F(t,x) \subset F(t,y) + k(t)\|x-y\|_X B \qquad \forall\, x, y \in X\,;$$

(A3) $\exists\, \phi \in L^1(I; \boldsymbol{R}^+)$ such that, for a.e. $t \in I$ and any $x \in X$,

$$F(t,x) \subset \phi(t)\,(1 + \|x\|_X)B\,.$$

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Denote by:

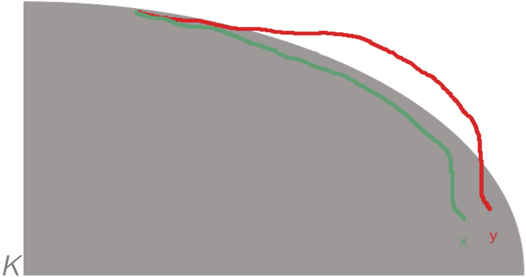
- $S_{[t_0,1]}(x_0)$ the set of solutions of (DI).
- $S^K_{[t_0,1]}(x_0)$ the set of solutions of (DI) - (SC).

Neighboring Feasible Trajectories

Definition

The problem (DI) - (SC) satisfies the Neighboring Feasible Trajectories [NFT] property if $\forall\, R > 0$, $\exists\, L > 0$ such that $\forall\, t_0 \in [0,1]$, $\forall\, x_0 \in K \cap RB$, $\forall\, y(\cdot) \in S_{[t_0,1]}(x_0)$, $\exists\, x(\cdot) \in S^K_{[t_0,1]}(x_0)$ satisfying

$$\|x(\cdot) - y(\cdot)\|_{\mathcal{C}([t_0,1],X)} \leq L \max_{t\in[t_0,1]} dist_K(y(t)).$$



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$$\|x(\cdot) - y(\cdot)\|_{\mathcal{C}([t_0,1],X)} \leq L \max_{t\in[t_0,1]} dist_K(y(t)).$$

Applications to finite dimensional optimal control problems:

- Regularity of the value function
- Dynamical programming
- Sensitivity relations
- Normality of the Pontryagin Maximum Principle
- Optimal synthesis
- Differential games

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$$\|x(\cdot) - y(\cdot)\|_{\mathcal{C}([t_0,1],X)} \leq L \max_{t\in[t_0,1]} dist_K(y(t)).$$

NFT Theorems have been proved in the finite dimensional case under different assumptions by several authors: P. Bettiol, A. Bressan, P. Cardaliaguet, G. Facchi, F. Forcellini, H. Frankowska, M. Quincampoix, F. Rampazzo, R. Vinter,...

Some notations

- given a non-empty closed $Q \subset X$,

$$d_Q(x) = \begin{cases} \inf_{q \in Q} \|x - q\|_X & \text{if } x \notin Q \\ -\inf_{q \in (X \setminus Q)} \|x - q\|_X & \text{otherwise} \end{cases}$$

  denotes the oriented distance from $x \in X$ to Q;
- for a Lipschitz continuous map $f : X \to \boldsymbol{R}$, let $\partial f(x)$ denote the Clarke generalized gradient at the point $x \in X$.
- for any $x \in X$, set

$$\sigma(x; y) = \sup_{\xi \in \partial d_K(x)} \langle \xi, y \rangle, \quad \forall y \in X .$$

Inward Pointing Condition. Finite dimension

$$\forall\, R > 0,\ \exists\, \rho > 0 \ \text{ such that } \ \forall \bar{x} \in \partial K \cap RB,$$

$$\text{if } \ \sigma(\bar{x}; v) \geq 0 \ \text{ for some } \ t \in I,\ v \in F(t, \bar{x}),$$

$$\text{then } \ \inf_{w \in F(t,\bar{x})} \sigma(\bar{x}; w - v) \leq -\rho.$$

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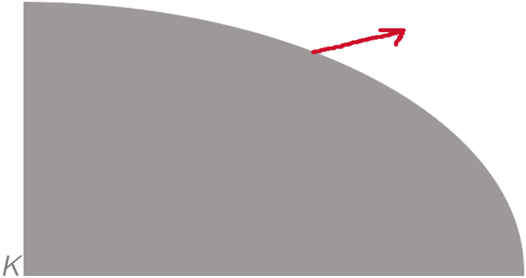

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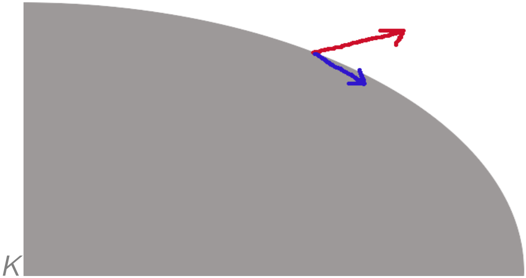

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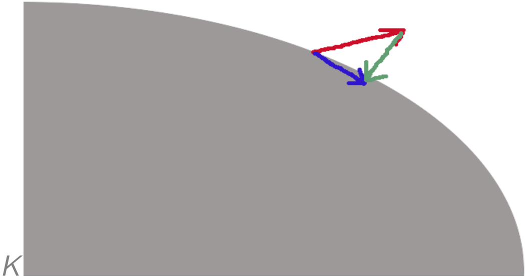

Inward Pointing Condition. Finite dimension

$\forall\, R > 0,\ \exists\, \rho > 0$ such that $\forall \bar{x} \in \partial K \cap RB$,

if $\sigma(\bar{x}; v) \geq 0$ for some $t \in I,\ v \in F(t, \bar{x})$,

then $\inf_{w \in F(t,\bar{x})} \sigma(\bar{x}; w - v) \leq -\rho.$

Remark: If ∂K is of class C^1, then the above condition is equivalent to the Soner inward pointing condition:
$\forall\, R > 0,\ \exists\, \rho > 0$ such that

$$\inf_{w \in F(t,\bar{x})} \langle \nabla d_K(\bar{x}), w \rangle \leq -\rho \quad \forall\, (t, \bar{x}) \in I \times (\partial K \cap RB).$$

Inward pointing condition (IPC). Infinite dimension

For any $R > 0$, there exist $\eta, \rho > 0$ such that for any $t \in I$, any $x \in RB \cap \partial^\eta K$, any $v \in F(t,x)$ satisfying

$$\Sigma_\eta(x; v) \geq 0,$$

there exists $w \in F(t,x)$ satisfying

$$\max\left\{\Sigma_\eta(x; w - v); \Sigma_\eta(x; w)\right\} \leq -\rho\,.$$

where, for any $\eta > 0$ and $x \in X$, we set

- $\partial^\eta K = \left\{x \in K + \eta B \,:\, S(\tau)\,x \in \partial K + \eta B \text{ for some } \tau \in [0,\eta]\right\}$,
- $\mathcal{A}^\eta(x) = \left\{(\tau, z) \in [0,\eta] \times X \,:\, S(\tau)\,x \in \partial K + \eta B,\, z \in B(S(\tau)x, \eta)\right\}$,
- $\Sigma_\eta(x; \cdot) : X \to [-\infty, +\infty) : v \mapsto \sup_{(\tau,z)\in\mathcal{A}^\eta(x)} \sigma(z; S(\tau)\, v)$.

Inward pointing condition (IPC). Infinite dimension

If some compactness is present, then the inward pointing condition can be replaced by the following "finite dimensional" version:

$$\forall\, R > 0, \exists\, \rho > 0 \ \text{ such that } \ \forall \bar{x} \in \partial K \cap RB,$$
$$\text{if } \ \sigma(\bar{x}; v) \geq 0 \ \text{ for some } \ t \in I,\, v \in F(t, \bar{x}), \ \text{ then}$$
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Examples: (1) Suppose that $F(\cdot,\bar{x})$ is continuous for all $\bar{x}\in X$ and that

$$F(t,\bar{x}) \ \text{ is compact for any } t\in I \text{ and } \bar{x}\in\partial K.$$

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For instance:

$$\begin{cases} x'(t) = \mathbb{A}x(t) + \sum_{i=1}^{m} u_i(t) f_i(x(t)) & \text{a.e. } t \in [t_0, 1] \\ u(t) \in \bar{J},\ J \subset I\,. \end{cases}$$

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$$\inf_{w \in F(t,\bar{x})} \sigma(\bar{x}; w - v) \leq -\rho.$$

Examples: (2) Suppose that $F(\cdot, \bar{x})$ is continuous for all $\bar{x} \in \partial X$ and that

- X is a reflexive space
- $F(t, \bar{x})$ is convex, for any $t \in I$ and $\bar{x} \in \partial K$;
- for any $t \in I$ and $\bar{x} \in \partial K$ the set-valued map $x \rightsquigarrow \partial d_K(\cdot)$ is upper semicontinuous at $\bar{x}$ and $\partial d_K(\bar{x})$ is compact.

Neighboring Feasible Trajectories. Infinite dimension

We say that K is invariant by the semigroup $S(t)$ if

$$S(t)\,K \subset K \qquad \forall\, t \in I\,.$$

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Theorem

Assume (A), (IPC) and that K is invariant by the semigroup $S(t)$. Then the problem (DI) - (SC) satisfies the [NFT] property.

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Assume (A), (IPC) and that K is invariant by the semigroup $S(t)$. Then the problem (DI) - (SC) satisfies the [NFT] property.

Corollary

Under the same assumptions, given a locally Lipschitz continuous cost function $g : X \to \boldsymbol{R}$, the corresponding value function

$$V : I \times X \to \boldsymbol{R} \cup \{+\infty\} : (t_0, y_0) \mapsto \inf\left\{g(x(1)) : x \in \mathcal{S}^K_{[t_0,1]}(y_0)\right\},$$

is continuous on $I \times K$ and the map $y_0 \mapsto V(t_0, y_0)$ is locally Lipschitz continuous on K uniformly in time.

Outline of the talk

Distance Estimates to Feasible Trajectories

Examples

Necessary Optimality Conditions

(1) A diffusion equation

$$\begin{cases} \partial_t u \in \Delta u + F(t,u) & \text{in } (0,1) \times \Omega \\ \partial_\nu u = 0 & \text{on } (0,1) \times \partial\Omega, \end{cases}$$

where $u = u(t, \mathbf{x})$ and Ω is a bounded domain of $\boldsymbol{R}^N$ with smooth boundary.

State constraint:

$$K = \left\{ u \in \mathcal{C}(\overline{\Omega}) : -1 \leq u(\mathbf{x}) \leq 1, \ \mathbf{x} \in \overline{\Omega} \right\}.$$

The operator associated to the system generates a C_0 semigroup in $X = \mathcal{C}(\overline{\Omega})$.

(2) A one-dimensional heat equation

Consider the differential inclusion in $X = H^1(0,1)$

$$\dot{x}(t,s) \in \partial_{ss}x(t,s) - x(t,s) + F(t,x(t,s)) \qquad (t,s) \in [0,1] \times [0,1],$$

endowed with the Neumann boundary condition.

The set $K = \{x \in X \ : \ x \geq 0\}$ is invariant by the semigroup generated by the associated linear operator.

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endowed with the Neumann boundary condition.

The set $K = \{x \in X \ : \ x \geq 0\}$ is invariant by the semigroup generated by the associated linear operator.

Here, the set $\{\xi \in \partial d_K(\bar{x}) \,|\, \bar{x} \in \partial K\}$ is pre-compact !!

The "finite dimensional" version of the inward pointing condition can be considered.

(3) A model for Boltzmann viscoelasticity[1]

Given an elastic body occupying a region $\Omega \subset \boldsymbol{R}^3$, a bounded domain with smooth boundary $\partial\Omega$, consider the inclusion

$$\begin{cases} \ddot{x}(t) - \Delta\left[x(t) - \int_0^\infty \mu(s)x(t-s)\,\mathrm{d}s\right] \in F(t, x(t)), & t > 0 \\ x(0) = x_0\,, \quad \dot{x}(0) = y_0\,, \quad x(-s)|_{s>0} = \varphi_0(s) \\ x(t)_{|\partial\Omega} = 0\,, \end{cases}$$

with $x_0 \in H_0^1(\Omega), y_0 \in L^2(\Omega), \varphi_0 \in L^2_\mu(\boldsymbol{R}^+; H_0^1(\Omega))$.

The memory kernel μ is a nonnegative nonincreasing and summable function on $\boldsymbol{R}^+$, with total mass $\kappa = \int_0^\infty \mu(s)\mathrm{d}s \in (0,1)$.

[1]P. Cannarsa, H. Frankowska and E.M. Marchini, Optimal control for evolution equations with memory, J. Evol. Equ., 2013

A model for Boltzmann viscoelasticity

Introducing the auxiliary variable

$$\eta = \eta^t(s) = x(t) - x(t-s), \qquad t \geq 0,\ s > 0\,,$$

we recast the problem as the system

$$\begin{cases} \ddot{x}(t) - \Delta\Big[(1-\kappa)x(t) + \displaystyle\int_0^\infty \mu(s)\eta^t(s)\mathrm{d}s\Big] \in F(t, x(t)), \\ \dot{\eta} = -\eta' + \dot{x}(t), \end{cases}$$

with

$$x(0) = x_0, \quad \dot{x}(0) = y_0, \quad \eta^0(s) = x_0 - \varphi_0(s),$$

in the product space $X = L^2(\Omega) \times H_0^1(\Omega) \times L^2_\mu(\mathbb{R}^+; H_0^1(\Omega))$.

The above operator generates a C_0-semigroup of contractions[2].

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[2]C.M. Dafermos, Asymptotic stability in viscoelasticity, Arch. Ration. Mech. Anal., 1970

Outline of the talk

Constrained Mayer problem

We consider the Mayer Problem

(MP) $$\text{minimize} \quad g(x(1))$$

over the set of trajectories $x : I \to X$ satisfying

(CS) $$\begin{cases} \dot{x}(t) = \mathbb{A}\, x(t) + f(t, x(t), u(t)) & \text{a.e. } t \in I\,, \\ x(0) \in Q_0\,, \quad x(t) \in K & \forall\, t \in I\,. \end{cases}$$

- X an infinite dimensional separable Banach space, Z a complete separable metric space;
- u is a measurable selection of $U : I \rightsquigarrow Z$;
- $\mathbb{A}$ is the infinitesimal generator of a strongly continuous semigroup $S(t) : X \to X$;
- $f : I \times X \times Z \to X$, $g : X \to \boldsymbol{R}$;
- Q_0 and K are nonempty closed subset of X.

Assumptions (H)

- $U : I \rightsquigarrow Z$ is measurable with nonempty closed values;
  f is measurable in t, Fréchet differentiable in x and continuous in u;
  g is Fréchet differentiable;
- $F : (t, x) \rightsquigarrow f(t, x, U(t))$ satisfies assumptions (A);
- K is invariant with respect to the semigroup $S(t)$;
- K and $\overline{\mathrm{co}}F$ satisfy (IPC).

Tangent cones

- the contingent cone to $Q \subset X$ at $x \in Q$ is defined by

$$T_Q(x) = \left\{ v \in X : \lim \inf_{h \to 0^+} \frac{dist_Q(x + hv)}{h} = 0 \right\};$$

- the Clarke tangent cone to Q at $x \in Q$ is defined by

$$C_Q(x) = \left\{ v \in X : \lim_{h \to 0^+, x' \to_Q x} \frac{dist_Q(x' + hv)}{h} = 0 \right\}.$$

A Fermat's Rule

Given an optimal pair $(\bar{x}, \bar{u})$ for (MP), set

$$\mathcal{T}(t) = T_{\overline{\mathrm{co}} f(t,\bar{x}(t),U(t))}(f(t,\bar{x}(t),\bar{u}(t))) \;\; \forall\, t \in I\,,$$

$$\mathcal{D} = \left\{ w \in \mathcal{C}(I,X) \;:\; w(t) \in C_K(\bar{x}(t))\,,\; \forall\, t \in I \right\}.$$

Theorem

Assume (H). Then every solution of

$$\begin{cases} \dot{w}(t) \in \mathbb{A}\, w(t) + \partial_x f(t,\bar{x}(t),\bar{u}(t))\, w(t) + \mathcal{T}(t) \quad \text{a.e. } t \in I\,, \\ w(0) \in T_{K \cap Q_0}(\bar{x}(0)), \end{cases}$$

belonging to $\mathcal{D}$, satisfies

$$\langle \nabla g(\bar{x}(1)), w(1) \rangle \geq 0.$$

A constrained Maximum Principle

Given an optimal pair $(\bar{x}, \bar{u})$ for (MP), let $\mathcal{S}$ be the solution operator associated to

$$\dot{w}(t) = \mathbb{A}\, w(t) + \partial_x f(t, \bar{x}(t), \bar{u}(t))\, w(t)\,.$$

Theorem

Assume (H) and and let C_0 be any closed convex cone contained in $T_{K\cap Q_0}(\bar{x}(0))$. Then, there exist $\lambda \in \{0,1\}$ and a measure $\gamma \in \mathcal{D}^-$ such that the function $z : I \to X^*$ defined by

$$z(s) = \mathcal{S}(1,s)^*\lambda\nabla g(\bar{x}(1)) + \int_s^1 \mathcal{S}(t,s)^*\gamma(\mathrm{d}t)$$

satisfies

$$\langle z(t), f(t, \bar{x}(t), \bar{u}(t))\rangle = \min_{u\in U(t)} \langle z(t), f(t, \bar{x}(t), u)\rangle \quad \text{for a.e. } t \in I$$

$$z(0) \in -C_0^-\,.$$

Normality of the Maximum Principle

For every $x \in K$ we introduce the cone

$$D_K(x) = \begin{cases} \{v \in X : \sigma(x;v) < 0\} & \text{if } x \in \partial K \\ X & \text{otherwise}. \end{cases}$$

Lemma

Assume (H). Then $\forall\, w_0 \in D_K(\bar{x}(0))$ there exists a solution of

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that belongs to Int $\mathcal{D}$.

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that belongs to Int $\mathcal{D}$.

Theorem (...continuation)

If $D_K(\bar{x}(0)) \cap C_0 \neq \emptyset$, we can set $\lambda = 1$ in the maximum principle.