Energy decay estimates for abstract evolution equations with time delay

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On the occasion of the 60th birthday of Piermarco Cannarsa

Locally damped wave

Let Ω be an open bounded domain of $\mathbb{R}^n$, $n \geq 1$, with a boundary $\partial\Omega$ of class C^2. We denote by $\nu(x)$ the outer unit normal vector at a point $x \in \partial\Omega$. Let m be the standard multiplier, $m(x) = x - x_0$, $x_0 \in \mathbb{R}^n$, and let ω be the intersection between an open neighborhood of the set

$$\Gamma_0 = \{\ x \in \partial\Omega\ :\ m(x) \cdot \nu(x) > 0\ \},$$

and Ω.
It is well-known that the problem

$$\begin{array}{l} u_{tt}(x,t) - \Delta u(x,t) + a\chi_\omega u_t(x,t) = 0, \quad \text{in} \ \ \Omega \times (0, +\infty), \\ u(x,t) = 0, \quad \text{on} \ \ \partial\Omega \times (0, +\infty) \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \text{in} \ \ \Omega, \end{array}$$

whith $a > 0$ and initial data $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, is exponentially stable, that is the energy

$$E(t) := \frac{1}{2} \int_\Omega [u_t^2(x,t) + |\nabla u(x,t)|^2] dx,$$

Locally damped wave

satisfies the uniform estimate,

$$E(t) \leq Ce^{-C't}E(0), \quad t > 0,$$

for all initial data.

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Exponential stability is not conserved in general in presence of a TIME DELAY!

Delay effects arise in many applications and practical problems and it is well-known that a delay arbitrarily small may destabilize a system which is uniformly asymptotically stable in absence of delay (see e.g. [Datko,Lagnese and Polis, 1986], [Datko, 1988]).

Locally damped wave with time delay

Let us consider the problem

$$
\begin{array}{l}
u_{tt}(x,t) - \Delta u(x,t) + a u_t(x, t-\tau) = 0, \quad \text{in} \;\; \Omega \times (0,+\infty), \\
u(x,t) = 0, \quad \text{on} \;\; \partial\Omega \times (0,+\infty), \\
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \text{in} \;\; \Omega, \\
u_t(x,t) = f(x,t), \quad \text{in} \;\; \Omega \times (-\tau, 0),
\end{array}
$$

where $\tau > 0$ is the time delay and the initial data are taken in suitable spaces.

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\end{aligned}$$

where $\tau > 0$ is the time delay and the initial data are taken in suitable spaces. In this case exponential stability FAILS!

Indeed, as shown in [Nicaise and P., 2006] it is possible to find for the above problem a sequence $\{\tau_k\}_k$ of delays with $\tau_k \to 0$ ($\tau_k \to \infty$) for which the corresponding solutions u_k have an increasing energy.

Stabilizing feedback

In [Nicaise and P., 2006] (cfr. [Xu, Yung and Li, 2006] for boundary delay in 1-d) in order to contrast the destabilizing effect of the time delay a "good" (not delayed) damping term is introduced in the first equation. More precisely the problem there considered is

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + a\chi_\omega u_t(x,t) + k\chi_\omega u_t(x,t-\tau) = 0, \quad x \in \Omega, \ t > 0, \\ u(x,t) = 0, \quad x \in \partial\Omega \ t > 0, \end{cases}$$

with $k, a \in \mathbb{R}, \ a > 0$, and initial data in suitable spaces. If $a > |k|$ the system is uniformly exponentially stable.

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CRUCIAL: The delayed term and the 'good' one are acting on the same set ω.

Delay feedback on a larger region

We want to analyze the case of 'good' damping acting on a smaller set than the delayed term.
Let us consider the system

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + a\chi_\omega u_t(x,t) + k u_t(x,t-\tau) = 0, \\ \qquad\qquad\qquad\qquad\qquad \text{in} \quad \Omega \times (0,+\infty), \\ u(x,t) = 0, \quad \text{on} \quad \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \text{in} \quad \Omega, \\ u_t(x,t) = f(x,t), \quad \text{in} \quad \Omega \times (-\tau, 0), \end{cases} \qquad (\mathbf{W})$$

where a, k are real numbers, $a > 0$, the time delay $\tau > 0$ is a positive constant and the initial data are taken in suitable spaces.

For sake of simplicity we consider the delay term acting in the whole domain Ω. The estimate holds true with analogous proof when the delay term acts in a subdomain $\tilde{\omega}$.

Well-posedness

Let us introduce the function

$$z(x,\rho,t) = u_t(x,t-\tau\rho), \quad x \in \Omega,\ \rho \in (0,1),\ t > 0\,.$$

Then, the above problem is equivalent to

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + a\chi_\omega u_t(x,t) + kz(x,1,t) = 0, \quad \text{in} \ \ \Omega \times (0,+\infty), \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0 \quad \text{in} \ \ \Omega \times (0,1) \times (0,+\infty), \\ u(x,t) = 0, \quad \text{on} \ \ \partial\Omega \times (0,+\infty) \\ z(x,0,t) = u_t(x,t), \quad \text{on} \ \ \Omega \times (0,+\infty) \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \text{in} \ \ \Omega, \\ z(x,\rho,0) = f(x,-\rho\tau), \quad \text{in} \ \ \Omega \times (0,1). \end{cases}$$

Let us denote

$$U := (u, u_t, z)^T,$$

then

$$U' := (u_t, u_{tt}, z_t)^T = (u_t, \Delta u - a\chi_\omega u_t - kz(\cdot,1,\cdot), -\tau^{-1} z_\rho)^T\,.$$

Well-posedness

Hence, the problem can be rewritten as

$$\begin{cases} U' = \mathcal{A}U\,, \\ U(0) = (u_0, u_1, f(\cdot, -\cdot\tau))^T\,, \end{cases} \qquad \textbf{(PW)}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\begin{pmatrix} u \\ v \\ z \end{pmatrix} := \begin{pmatrix} v \\ \Delta u - a\chi_\omega v - kz(\cdot, 1) \\ -\tau^{-1} z_\rho \end{pmatrix},$$

with domain

$$\mathcal{D}(\mathcal{A}) := \Big\{ (u, v, z)^T \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \times L^2(\Omega; H^1(0,1)) \ : \ v = z(\cdot, 0) \ \text{ in } \ \Omega \Big\}.$$

Well-posedness

Denote by $\mathcal{H}$ the Hilbert space

$$\mathcal{H} := H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega \times (0,1)),$$

equipped with the inner product

$$\left\langle \begin{pmatrix} u \\ v \\ z \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{z} \end{pmatrix} \right\rangle := \int_\Omega \{\nabla u(x) \nabla \tilde{u}(x) + v(x)\tilde{v}(x)\} dx$$
$$+\xi\tau \int_\Omega \int_0^1 z(x,\rho)\tilde{z}(x,\rho) d\rho dx,$$

where ξ is any fixed positive number.

Proposition [P., 2012]

For any initial datum $U_0 \in \mathcal{H}$ *there exists a unique solution* $U \in C([0,+\infty), \mathcal{H})$ *of problem* (**PW**). *Moreover, if* $U_0 \in \mathcal{D}(\mathcal{A})$, *then*

$$U \in C([0,+\infty), \mathcal{D}(\mathcal{A})) \cap C^1([0,+\infty), \mathcal{H}).$$

Stability result

Let us introduce the energy of any solution of problem (**W**) as

$$E(t) := E(u,t) = \frac{1}{2}\int_\Omega \{u_t^2(x,t) + |\nabla u(x,t)|^2\}dx \\ + \frac{\xi}{2}\int_{t-\tau}^t \int_\Omega u_t^2(x,s)dsdx,$$

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where ξ is a suitable positive constant.
By differentiating the energy,

$$E'(t) \le -\left(a - \frac{|k|+\xi}{2}\right)\int_\omega u_t^2(x,t)dx - \frac{\xi-|k|}{2}\int_\Omega u_t^2(x,t-\tau)dx \\ + \frac{\xi+|k|}{2}\int_{\Omega\setminus\omega} u_t^2(x,t)dx.$$

Then, choose ξ in the definition of the energy such that

$$|k| < \xi < a\,. \qquad (\mathbf{H}\xi)$$

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Assuming this and choosing ξ as before, we observe that the coefficients of the first two integrals are negative. On the contrary, it remains a positive contribution related to the subdomain $\Omega \setminus \omega$ where only the delayed damping acts.

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THEOREM [P., 2012]

There exists $k_0 \in (0, a)$ such that for every k with $|k| < k_0$, there are positive constants K, μ for which

$$E(t) \leq Ke^{-\mu t}E(0), \quad t > 0,$$

for every solution of $(\mathbf{W})$.

The auxiliary problem

In order to overcome the difficulty due to the fact that the energy is not decreasing, we introduce the auxiliary problem

$$\left\{\begin{array}{l} \varphi_{tt}(x,t) - \Delta\varphi(x,t) + a\chi_\omega\varphi_t(x,t) + k\varphi_t(x,t-\tau) + \xi\chi_{\Omega\setminus\omega}\varphi_t(x,t) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{in } \Omega\times(0,+\infty), \\ \varphi(x,t) = 0, \quad \text{on} \quad \partial\Omega\times(0,+\infty), \\ \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), \quad \text{in} \quad \Omega, \\ \varphi_t(x,t) = g(x,t), \quad \text{in} \quad \Omega\times(-\tau,0), \end{array}\right. \qquad (\tilde{\mathbf{W}})$$

For solutions of problem $(\tilde{\mathbf{W}})$ the energy $F(\cdot)$,

$$F(t) := F(\varphi,t) = \frac{1}{2}\int_\Omega \{\varphi_t^2(x,t) + |\nabla\varphi(x,t)|^2\}dx + \frac{\xi}{2}\int_{t-\tau}^t\int_\Omega \varphi_t^2(x,s)dsdx,$$

with ξ satisfying $|k| < \xi < a$, is decreasing in time.

Stability of the auxiliary problem

$\Rightarrow$ We can prove an exponential stability result for the perturbed problem $(\tilde{\mathbf{W}})$, for all $|k| < a$.

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Now, we look at the former problem $(\mathbf{W})$

$$u_{tt}(x,t) - \Delta u(x,t) + a\chi_\omega u_t(x,t) + ku_t(x,t-\tau) = 0,$$

$$+\text{B.C. and I.C.}$$

as a bounded perturbation of the auxiliary one $(\tilde{\mathbf{W}})$

$$\varphi_{tt}(x,t) - \Delta\varphi(x,t) + a\chi_\omega\varphi_t(x,t) + k\varphi_t(x,t-\tau) + \xi\chi_{\Omega\setminus\omega}\varphi_t(x,t) = 0,$$

$$+\text{B.C. and I.C.}$$

Stability result

Now,

Theorem [Pazy] Let X be a Banach space and let A be the infinitesimal generator of a C_0 semigroup $T(t)$ on X, satisfying $\|T(t)\| \leq Me^{\omega t}$. If B is a bounded linear operator on X then $A + B$ is the infinitesimal generator of a C_0 semigroup $S(t)$ on X, satisfying $\|S(t)\| \leq Me^{(\omega + M\|B\|)t}$.

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careful estimates on the involved constants

$\Longrightarrow$ exponential stability of problem (**W**) when the coefficient $|k|$ satisfies a suitable smallness condition.

Abstract second order evolution equations

Let H be a real Hilbert space with norm and inner product $\|\cdot\|_H$ and $\langle\cdot,\cdot\rangle_H$ and let

$$A : \mathcal{D}(A) \to H$$

be a positive self–adjoint operator with a compact inverse in H. Denote by $V := \mathcal{D}(A^{\frac{1}{2}})$. Moreover, for $i = 1, 2$, let U_i be real Hilbert spaces with norm and inner product $\|\cdot\|_{U_i}$ and $\langle\cdot,\cdot\rangle_{U_i}$ and let

$$B_i : U_i \to V'$$

be linear operators. In this setting we consider the problem

$$\begin{array}{ll} u_{tt}(t) + Au(t) + B_1 B_1^* u_t(t) + B_2 B_2^* u_t(t-\tau) = 0 \quad t > 0, & \\ u(0) = u_0 \quad \text{and} \quad u_t(0) = u_1, & \textbf{(P)} \\ B_2^* u_t(t) = f^0(t) \quad t \in (-\tau, 0), & \end{array}$$

where the constant $\tau > 0$ is the time delay. We assume that the delay feedback operator B_2 is bounded, that is $B_2 \in \mathcal{L}(U_2, H)$, while the standard one $B_1 \in \mathcal{L}(U_1, V')$ may be unbounded.

Abstract second order evolution equations

Inspired to previous result for locally damped wave, we are interested in giving stability results for the above problem under a suitable assumption on the size of the feedback operator B_2, when the feedback B_1 is a stabilizing one.
Assuming that an observability inequality holds for $(\mathbf{P})$ when $B_2 = 0$, through the definition of a suitable energy we obtain sufficient conditions ensuring exponential stability [S. Nicaise & P., 2014].

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Our abstract framework includes several concrete examples.

Wave equation with internal and boundary dampings

Let $\Omega \subset \mathbb{R}$ be an open bounded domain with a Lipschitz boundary $\partial\Omega$. We assume $\partial\Omega = \Gamma_0 \cup \Gamma_1$, where $\Gamma_0,\ \Gamma_1$ are closed subsets of $\partial\Omega$ with $\Gamma_0 \cap \Gamma_1 = \emptyset$. Moreover we assume that Γ_0 and Γ_1 have an non empty interior (on $\partial\Omega$).
Given $k \in L^\infty(\Gamma_0)$ and $b \in L^\infty(\Omega)$ such that $b(x) \geq 0$ a.e. $x \in \Omega$ and

$$k(x) \geq k_0 > 0 \text{ a.e. } x \in \Gamma_0,$$

then the problem

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + b(x)u_t(x,t-\tau) = 0, \quad x \in \Omega,\ t > 0 \\ u(x,t) = 0, \quad x \in \Gamma_1,\ t > 0 \\ \frac{\partial u}{\partial \nu}(x,t) = -k(x)u_t(x,t), \quad x \in \Gamma_0,\ t > 0 \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \quad x \in \Omega \\ u_t(x,t) = f^0(x,t), \quad x \in \Omega,\ t \in (-\tau,0) \end{cases} \qquad (\mathbf{WB})$$

with initial data in a suitable space, enters in the abstract framework.

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u(x,t) = 0, \quad x \in \Gamma_1,\ t > 0 & (\mathbf{WB}) \\
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\end{array}$$

with initial data in a suitable space, enters in the abstract framework.

$\Longrightarrow$ The stability result applies if $\|b\|_\infty$ satisfies a suitable smallness condition.

Viscoelastic wave equation

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with a smooth boundary.

$$\begin{aligned} & u_{tt}(x,t) - \Delta u(x,t) + \int_0^\infty \mu(s)\Delta u(x,t-s)ds \\ & \qquad\qquad\qquad\qquad + k u_t(x,t-\tau) = 0 \quad \text{in } \Omega \times (0,+\infty) \\ & u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,+\infty) \qquad\qquad\qquad\qquad (\mathbf{M}) \\ & u(x,t) = u_0(x,t) \quad \text{in } \Omega \times (-\infty, 0] \end{aligned}$$

where k is a real number, $\tau > 0$ is the time delay, the initial datum u_0 belongs to a suitable space and the memory kernel $\mu : [0,+\infty) \to [0,+\infty)$ is a locally absolutely continuous function satisfying

i) $\mu(0) = \mu_0 > 0$;

ii) $\int_0^{+\infty} \mu(t)dt = \tilde{\mu} < 1$;

iii) $\mu'(t) \leq -\alpha\mu(t)$, for some $\alpha > 0$.

Memory against small delay feedback

We know that the above problem is exponentially stable for $k = 0$ (see e.g. [Giorgi, Munõz Rivera & Pata, 2001]). Since the viscoelastic damping is a stabilizing one, it is natural to investigate if it prevails with respect the time delay term.

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We can show [Alabau-Boussouira, Nicaise & P., 2014], using the perturbative approach introduced in [P., 2012] and adapting some multiplier estimates in [Alabau-Boussouira, Cannarsa & Sforza, 2008)], that even if a time delay generates instability, an exponential stability result still holds if the delay parameter k satisfies a suitable *smallness* condition.

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See also [P., 2016] for stability results for viscoelastic wave equation with intermittent delay feedback. Stability estimates are obtained under a condition on the L^1-norm of the (time-variable) coefficient $k(\cdot)$. Extension to $k = k_1 + k_2$, $k_1 \in L^1$ and $k_2 \in L^\infty$ in [Komornik & P., in preparation].

Abstract result

Let $\mathcal{H}$ be a fixed Hilbert space with norm $\|\cdot\|$, and consider an operator $\mathcal{A}$ from $\mathcal{H}$ into itself that generates a C_0-semigroup $(S(t))_{t\geq 0}$ that is exponentially stable, i.e., there exist two positive constants M and ω such that

$$\|S(t)\|_{\mathcal{L}(\mathcal{H})} \leq Me^{-\omega t}, \ \forall t \geq 0.$$

For a fixed delay parameter τ, a fixed bounded operator $\mathcal{B}$ from $\mathcal{H}$ into itself and for a real parameter k, we consider the evolution equation

$$\begin{cases} U_t(t) = \mathcal{A}U(t) + F(U(t)) + k\mathcal{B}U(t-\tau) \quad \text{in } (0,+\infty) \\ U(0) = U_0, \ \mathcal{B}U(t-\tau) = f(t), \quad \forall t \in (0,\tau), \end{cases} \qquad \textbf{(PA)}$$

where $F : \mathcal{H} \to \mathcal{H}$ satisfies some Lipschitz conditions, the initial datum U_0 belongs to $\mathcal{H}$ and $f \in C([0,\tau];\mathcal{H})$.

Abstract result

Motivated by previous examples we are interested in giving an exponential stability result for such a problem under a suitable condition between the constant k and the constants M, ω, τ, the norm of $\mathcal{B}$ and the nonlinear term F.

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For particular examples (see e.g. [Bátkai & Piazzera (2005), Ammari, Nicaise & P. (2010), P. (2012), Said-Houari & Soufyane (2012), Alabau-Boussouira, Nicaise & P. (2014), Dai & Yang (2014), (2016)]) we know that the above problem, under certain *smallness* conditions on the delay feedback $k\mathcal{B}$, is exponentially stable, the proof being from time to time quite technical because some observability inequalities or perturbation methods are used.

Abstract result

Hence our main goal is to furnish a direct proof of this stability result by using the so-called Duhamel's formula (or variation of parameters formula).

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Note that our proof is simpler with respect to the ones used so far for particular models. Moreover, we emphasize its generality. Indeed, it applies to every model in the previous abstract form when the operator $\mathcal{A}$ generates an exponentially stable semigroup.

Well-posedness

Now, we assume that F is globally Lipschitz continuous, namely

$$\exists \gamma > 0 \quad \text{such that} \quad \|F(U_1) - F(U_2)\| \leq \gamma \|U_1 - U_2\|, \quad \forall\ U_1, U_2 \in \mathcal{H}\,.$$

Moreover, we assume that $F(0) = 0$.
The following well–posedness result holds.

Proposition

For any initial datum $U_0 \in \mathcal{H}$ and $f \in C([0, \tau]; \mathcal{H})$, there exists a unique (mild) solution $U \in C([0, +\infty), \mathcal{H})$ of problem (PA). Moreover,

$$U(t) = S(t)U_0 + \int_0^t S(t-s)[F(U(s)) + k\mathcal{B}U(s-\tau)]\, ds. \qquad (D)$$

Well-posedness

Proof. We use an iterative argument. Namely in the interval $(0, \tau)$, problem (PA) can be seen as an inhomogeneous evolution problem

$$\begin{cases} U_t(t) = \mathcal{A}U(t) + F(U(t)) + g_0(t) \quad \text{in } (0, \tau) \\ U(0) = U_0, \end{cases}$$

where $g_0(t) = kf(t)$. This problem has a unique solution $U \in C([0, \tau], \mathcal{H})$ (see [Th. 1.2, Ch. 6 of Pazy (1983)]) satisfying

$$U(t) = S(t)U_0 + \int_0^t S(t-s)[F(U(s)) + g_0(s)]\, ds.$$

This yields $U(t)$, for $t \in [0, \tau]$.

Therefore on $(\tau, 2\tau)$, problem (PA) can be seen as an inhomogeneous evolution problem

$$\begin{cases} U_t(t) = \mathcal{A}U(t) + F(U(t)) + g_1(t) \quad \text{in } (\tau, 2\tau) \\ U(\tau) = U(\tau-), \end{cases}$$

where $g_1(t) = k\mathcal{B}U(t-\tau)$. Hence, this problem has a unique solution $U \in C([\tau, 2\tau], \mathcal{H})$ given by

$$U(t) = S(t-\tau)U(\tau-) + \int_\tau^t S(t-s)[F(U(s)) + g_1(s)]\, ds, \forall t \in [\tau, 2\tau].$$

By iteration, we obtain a global solution U satisfying (F). $\square$

The following exponential stability result holds.

THEOREM [Nicaise & P., 2015]

There is a positive constant k_0 such that for k satisfying

$$|k| < k_0 := \frac{e^{\tau\omega} - 1}{\tau \|\mathcal{B}\|_{\mathcal{L}(\mathcal{H})} M e^{\tau\omega}},$$

and for $\gamma < \gamma(|k|)$, where $\gamma(|k|)$ is a suitable constant depending on $|k|$, there exist $\omega' > 0$ and $M' > 0$ such that the solution $U \in C([0,+\infty), \mathcal{H})$ of problem (P), with $U_0 \in \mathcal{H}$ and $f \in C([0,\tau]; \mathcal{H})$, satisfies

$$\|U(t)\|_{\mathcal{H}} \leq M' e^{-\omega' t}(\|U_0\|_{\mathcal{H}} + \int_0^\tau e^{\omega s}\|f(s)\|_{\mathcal{H}}\, ds), \quad \forall t \geq \tau.$$

From its definition the constant k_0 depends only on M, ω, τ and the norm of $\mathcal{B}$.

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From its definition the constant k_0 depends only on M, ω, τ and the norm of $\mathcal{B}$.

The proof relies again on an iterative argument and uses the Duhamel's formula and the Gronwall's Lemma.

The case F locally Lipschitz

Now we consider that F is only locally Lipschitz. More precisely, we assume that for every constant c there exists a positive constant $L(c)$ such that

$$\|F(U_1) - F(U_2)\| \le L(c)\|U_1 - U_2\|\,,$$

for all $U_1, U_2 \in \mathcal{H}$ with $\|U_1\| \le c,\ \|U_2\| \le c$.
Moreover, we assume that there exists an increasing continuous function $\chi : [0, +\infty) \to [0, +\infty)$, with $\chi(0) = 0$, such that

$$\|F(U)\| \le \chi(\|U\|)\|U\|,\ \forall\ U \in \mathcal{H}\,.$$

We can give an exponential stability result under a well–posedness assumption for *small* initial data. This assumption is satisfied for a quite large class of examples.

The case F locally Lipschitz

THEOREM [Nicaise & P., 2015]
Suppose that for $|k|$ sufficiently small

$$\exists\ \rho_0 > 0 \text{ and } \ C_{\rho_0} > 0 \ \text{ such } \ \text{that}$$
$$\forall\ U_0 \in \mathcal{H}, f \in C([0,\tau];\mathcal{H}) \text{ with } (\|U_0\|^2 + \int_0^\tau |k| \|f(s)\|^2 ds)^{1/2} < \rho_0,$$
$$\text{there exists a unique global solution} \qquad (HW)$$
$$U \in C([0,+\infty,\mathcal{H}) \text{ with } \ \|U(t)\| \leq C_{\rho_0} < \chi^{-1}\left(\frac{\omega'}{\tilde{M}}\right), \forall\ t > 0,$$

where ω' and $\tilde{M}$ are the constants of the exponential decay estimate of the linear part of the model.
Then there exists $\tilde{k} > 0$ such that if $|k| < \tilde{k}$, for every $U_0 \in \mathcal{H}$, $f \in C([0,\tau];\mathcal{H})$ satisfying the assumption from (H), the solution U satisfies the exponential decay estimate

$$\|U(t)\|_{\mathcal{H}} \leq \tilde{M} e^{-\tilde{\omega} t}(\|U_0\|_{\mathcal{H}} + \int_0^\tau e^{\omega s} \|f(s)\|_{\mathcal{H}}\, ds), \quad \forall t \geq \tau,$$

More general

As before, let $\mathcal{H}$ be a fixed Hilbert space with norm $\|\cdot\|$, and consider an operator $\mathcal{A}$ from $\mathcal{H}$ into itself that generates a C_0-semigroup $(S(t))_{t\geq 0}$ that is exponentially stable, i.e., there exist two positive constants M and ω such that

$$\|S(t)\|_{\mathcal{L}(\mathcal{H})} \leq Me^{-\omega t}, \ \forall t \geq 0.$$

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We consider the evolution equation

$$\begin{cases} U_t(t) = \mathcal{A}U(t) + \sum_{i=1}^{l} F_i(U(t), U(t-\tau_i)) & \text{in } (0, +\infty) \\ U(t-\tau) = U_0(t), \quad \forall t \in (0, \tau], \end{cases} \qquad \textbf{(PN)}$$

where l is a positive natural number and $\tau_i > 0, \ i = 1, \cdots, l$, are time delays.

More general

Without loss of generality we can suppose that the delays are different from each other and that

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$$\tau_i < \tau = \tau_1, \ \forall i = 2, \cdots, I.$$

The nonlinear terms $F_i : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ satisfy some Lipschitz conditions, while the initial datum U_0 satisfies $U_0 \in C([0, \tau]; \mathcal{H})$.
We denote by U^0 the initial datum at $t = 0$, namely $U^0 = U_0(\tau) \in \mathcal{H}$.

F_i Lipschitz continuous

Let the functions F_i, $i = 1, \dots, I$, be globally Lipschitz continuous, namely for every $i = 1, \dots, I$,

$$\exists \gamma_i > 0 \quad \text{such that}$$
$$\|F_i(U_1, U_2) - F_i(U_1^*, U_2^*)\|_{\mathcal{H}} \le \gamma_i(\|U_1 - U_1^*\|_{\mathcal{H}} + \|U_2 - U_2^*\|_{\mathcal{H}}),$$
$$\forall\ (U_1, U_2), (U_1^*, U_2^*) \in \mathcal{H} \times \mathcal{H}\,.$$

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$$\forall\ (U_1, U_2), (U_1^*, U_2^*) \in \mathcal{H} \times \mathcal{H}\,.$$

The following well–posedness result holds.

Proposition [Nicaise & P.,2017]

For any initial datum $U_0 \in C([0, \tau]; \mathcal{H})$, there exists a unique (mild) solution $U \in C([0, +\infty), \mathcal{H})$ of problem (PN).
Moreover,

$$U(t) = S(t)U^0 + \int_0^t S(t-s) \sum_{i=1}^{I} F_i(U(s), U(s - \tau_i))\, ds.$$

F_i Lipschitz continuous: stability result

Theorem [Nicaise & P., 2017]

Assume that the functions F_i satisfy also $F_i(0,0) = 0$, for all $i = 1, \dots, I$. With M, ω as before, we assume that

$$\gamma = \sum_{i=1}^{I} \gamma_i < \frac{\omega}{2M}\,.$$

If the time delay τ satisfies the *smallness* condition

$$\tau < \tau_0 := \frac{1}{\omega} \ln\left(\frac{\omega}{M\gamma} - 1\right),$$

then there exists $\omega' > 0$ such that the solution $U \in C([0,+\infty), \mathcal{H})$ of problem (PN), with $U_0 \in C([0,\tau];\mathcal{H})$, satisfies

$$\|U(t)\|_{\mathcal{H}} \leq Me^{-\omega' t}\Big(\|U_0\|_{\mathcal{H}} + \sum_{i=1}^{I} \gamma_i \int_0^{\tau_i} e^{\omega s}\|U(s-\tau_i)\|_{\mathcal{H}}\, ds\Big), \quad \forall t \geq 0\,.$$

Example: damped wave equation

Let Ω be a bounded domain in $\mathbb{R}^n$ with a boundary Γ of class C^2. Let $f_j : \mathbb{R} \to \mathbb{R}$ be globally Lipschitz continuous functions, $j = 1, 2$, satisfying $f_1(0) = f_2(0) = 0$. Let us consider the following semilinear damped wave equation:

$$
\begin{aligned}
&u_{tt}(x,t) - \Delta u(x,t) + a(x)u_t(x,t) = f_1(u(x,t)) + f_2(u(x,t-\tau)) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in} \quad \Omega \times (0,+\infty), \\
&u(x,t) = 0 \quad \text{in} \quad \Omega \times (0,+\infty), \\
&u(x,t-\tau) = u_0(x,t), \quad u_t(x,t-\tau) = u_1(x,t) \quad \text{in } \Omega \times (0,\tau]\,,
\end{aligned}
$$

where $\tau > 0$ is the time delay and the damping coefficient $a \in L^\infty(\Omega)$ satisfies

$$a(x) \geq a_0 > 0, \quad \text{a.e.} \quad x \in \omega\,,$$

for some nonempty open subset ω of Ω satisfying some control geometric property (see e.g. [Bardos, Lebeau & Rauch (1992)]).

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Setting $U = (u, u_t)^T$, this problem can be rewritten in abstract form with $\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega)$,

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ \Delta & -a \end{pmatrix}$$

and $F_1(U(t), U(t-\tau)) = (0, f_1(u(t)) + f_2(u(t-\tau)))^T$.

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and $F_1(U(t), U(t-\tau)) = (0, f_1(u(t)) + f_2(u(t-\tau)))^T$.

It is well–known that $\mathcal{A}$ generates a strongly continuous semigroup which is exponentially stable (see e.g. [Zuazua (1990)]), thus the assumptions on f_1, f_2 ensure that the well–posedness and the stability results apply to this model giving an exponential decay estimate of the energy for small values of the time delay τ.

More general nonlinearities

We consider a more general class of nonlinearities but assume that $\mathcal{A}$ is a negative selfadjoint operator in $\mathcal{H}$.

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In this case, $\mathcal{A}$ generates an analytic semigroup and existence results for problem (PN) can be obtained for nonlinear terms satisfying the next hypothesis (H1).

More precisely, we recall that $\mathcal{V} = D((-\mathcal{A})^{\frac{1}{2}})$ is a Hilbert space with the norm

$$\|U\|_{\mathcal{V}}^2 = ((-\mathcal{A})^{\frac{1}{2}} U, (-\mathcal{A})^{\frac{1}{2}} U), \ \forall\ U \in \mathcal{V}.$$

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$$\|U\|_{\mathcal{V}}^2 = ((-\mathcal{A})^{\frac{1}{2}} U, (-\mathcal{A})^{\frac{1}{2}} U), \ \forall \ U \in \mathcal{V}.$$

Furthermore if λ_1 is the smallest eigenvalue of $-\mathcal{A}$, we have

$$\lambda_1 \|U\|_{\mathcal{H}}^2 \leq \|U\|_{\mathcal{V}}^2, \quad \forall \ U \in \mathcal{V}.$$

More general nonlinearities

We assume that there exist a positive real number $\beta < \frac{1}{2}$, a constant C_0 and two continuous functions h_1, h_2 from $[0, \infty)^4$ to $[0, \infty)$ such that, for all $i = 1, \dots, l$,

$$\begin{aligned} \|F_i(U_1, V_1) - F_i(U_2, V_2)\|_{\mathcal{H}} \leq C_0 \|U_1 - U_2\|_{\mathcal{H}} \\ + h_1(\|U_1\|_{\mathcal{V}}, \|U_2\|_{\mathcal{V}}, \|V_1\|_{\mathcal{V}}, \|V_2\|_{\mathcal{V}}) \|U_1 - U_2\|_{\mathcal{V}} \\ + h_2(\|U_1\|_{\mathcal{V}}, \|U_2\|_{\mathcal{V}}, \|V_1\|_{\mathcal{V}}, \|V_2\|_{\mathcal{V}}) \|V_1 - V_2\|_{D((-\mathcal{A})^\beta)}, \end{aligned} \tag{H1}$$

for all $(U_1, U_2), (V_1, V_2) \in \mathcal{V} \times \mathcal{V}$.

Well-posedness

The following local existence result holds.

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Proposition [Nicaise & P., 2017]

Assume that the functions F_i satisfy (H1). Then for any initial datum $U_0 \in C([0,\tau];\mathcal{V}) \cap C^{0,\theta}([0,\tau], D((-\mathcal{A})^{\beta}))$, with $\beta < \frac{1}{2}$ from the assumption $(H1)$ and $\theta = \min\{\beta, \frac{1}{2} - \beta\}$, there exist a time $T_\infty \in (0, +\infty]$ and a unique solution

$$U \in C([0, T_\infty), \mathcal{V}) \cap C^1((0, T_\infty), \mathcal{H})$$

of problem (PN). If $T_\infty < +\infty$, then

$$\lim_{t \to T_\infty^-} \|U(t)\|_{\mathcal{V}} = +\infty.$$

Exponential stability for small data

We now give an exponential stability result for *small* initial data.

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For that purpose, we need the additional assumption on our nonlinear functions, namely we suppose that there exist a positive constant C_1 and a continuous function h_3 from $[0,\infty)^{I+1}$ to $[0,\infty)$ satisfying $h_3(0)=0$ and such that

$$\Big|\sum_{i=1}^{I}(W, F_i(U, V_i))_{\mathcal{H}}\Big| \\ \le \|W\|_{\mathcal{H}}\,(C_1\|U\|_{\mathcal{H}} + h_3(\|U\|_{\mathcal{V}}, \|V_1\|_{\mathcal{V}}, \cdots, \|V_I\|_{\mathcal{V}})\|U\|_{\mathcal{V}}), \quad (H2)$$

for all $W\in\mathcal{H}$, $U, V_i\in\mathcal{V}$, $i=1,\cdots,I$.

Exponential stability for small data

Theorem [Nicaise & P., 2017]

Assume that $(H1)$ and $(H2)$ are satisfied. With C_1 from the assumption $(H2)$, we assume that

$$\frac{C_1}{\lambda_1} - 1 < 0.$$

Then there exist $K_0 > 0$ small enough and $\gamma_0 < 1$ (depending on K_0) such that for all $K \in (0, K_0]$ and $U_0 \in C([0, \tau]; \mathcal{V})$ satisfying

$$\|U_0(t)\|_{\mathcal{V}} < \gamma_0 K, \forall\ t \in [-\tau, 0],$$

problem (PN) has a global solution U that satisfies the exponential decay estimate

$$\|U(t)\|_{\mathcal{H}} \leq M e^{-\tilde{\omega} t} \quad \forall t \geq 0\,,$$

for a positive constant M depending on U_0 and a suitable positive constant $\tilde{\omega}$.

Delay diffusion equations

We consider the semilinear diffusion equation with time delay

$$
\begin{aligned}
& u_t(t) - \Delta u(t) = f(u(t), u(t-\tau)) \quad \text{in } \Omega \times (0, +\infty), \\
& Bu(x,t) = 0 \quad \text{on } \Gamma \times (0, +\infty), \qquad\qquad (DD) \\
& u(x,t)) = u_0(x,t) \quad \text{in } \Omega \times [-\tau, 0],
\end{aligned}
$$

where the constant $\tau > 0$ is the time delay and the initial datum u_0 belongs to the space $C([0,\tau]; L^2(\Omega))$. The operator B is in the form

$$Bu = \alpha \partial_n u + \alpha' u,$$

with either $\alpha = 0$ and $\alpha' = 1$ corresponding to the case of Dirichlet boundary conditions or $\alpha = 1$ and $\alpha' \geq 0$ (with $\alpha' \in L^\infty(\partial\Omega)$) corresponding to the case of Neumann–Robin boundary conditions.

Delay diffusion equations

For further uses, in the case of Neumann conditions ($\alpha = 1, \alpha' = 0$), we fix a positive real parameter ε (that may depend on f), otherwise we set $\varepsilon = 0$.

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Now we assume that the nonlinearity $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies the following assumption:

there exist a non-negative constant α_0 (that may depend on ε) and two polynomials P_1 and P_2 (of one real variable) of degree n_1 and n_2 in the form

$$P_i(X) = \sum_{j=1}^{n_i} \alpha_{i,j} X^j,$$

with non negative real numbers $\alpha_{i,j}$ such that

$$\begin{aligned} &|f(x_1, y_1) - f(x_2, y_2) + \varepsilon(x_1 - x_2)| \\ &\qquad \leq \alpha_0 |x_1 - x_2| + P_1(|x_1| + |y_1| + |x_2| + |y_2|)|x_1 - x_2| \\ &\qquad + P_2(|x_1| + |y_1| + |x_2| + |y_2|)|y_1 - y_2|, \forall (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2. \end{aligned} \tag{HD1}$$

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In particular this assumption means that f is only locally Lipschitz.

Delay diffusion equations

Under this assumption, problem (DD) enters in the previous abstract framework.

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$$D(\mathcal{A}) := \{u \in H^1(\Omega) : \Delta u \in L^2(\Omega) \text{ and satisfying } Bu = 0 \text{ on } \Gamma\},$$

and

$$\mathcal{A}u = \Delta u - \varepsilon u, \quad \forall\, u \in D(\mathcal{A}).$$

Delay diffusion equations

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Note that for any $u \in \{u \in H^1(\Omega) : \Delta u \in L^2(\Omega)\}$, Bu has a meaning (as element of $H^{-\frac{1}{2}}(\Gamma)$ if $\alpha > 0$). It is not difficult to show that $-\mathcal{A}$ is a positive selfadjoint operator in $\mathcal{H}$ since it is the Friedrichs extension of the triple $(\mathcal{H}, \mathcal{V}, a)$ where $\mathcal{V} = H_0^1(\Omega)$ in case of Dirichlet boundary conditions otherwise $\mathcal{V} = H^1(\Omega)$, and the sesquilinear form a is defined by

$$a(u, v) = \int_\Omega (\nabla u \cdot \nabla \bar{v} + \varepsilon u \cdot \bar{v})\, dx + \int_\Gamma \alpha' u \cdot \bar{v}\, d\sigma(x),$$

that is symmetric, continuous and strongly coercive on $\mathcal{V}$.

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Let us notice that in the case of Neumann boundary conditions, the smallest eigenvalue λ_1 of $-\mathcal{A}$ is ε, otherwise it does not depend on ε and corresponds to the smallest eigenvalue of the Dirichlet problem or to the Robin eigenvalue problem.

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By introducing the function

$$F_1(x, y) = f(x, y) + \varepsilon x,$$

we see that problem (DD) can be written as (PN) with $I = 1$, $U(t) = u(\cdot, t)$ and $U_0(t) = u_0(\cdot, t)$.

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we see that problem (DD) can be written as (PN) with $l = 1$, $U(t) = u(\cdot, t)$ and $U_0(t) = u_0(\cdot, t)$.

Consequently the next local existence result follows from the abstract result.

Delay diffusion equations

Proposition [S. Nicaise e P., 2017]

Assume that $(HD1)$ holds and that d satisfies

$$d \le 2\Big(1 + \frac{1}{n_1}\Big) \quad \text{and} \quad d < 2\Big(1 + \frac{1}{n_2}\Big). \qquad (HD2)$$

Then there exists $\beta \in (0, \frac{1}{2})$ such that for any initial datum

$$u_0 \in C([0,\tau]; \mathcal{V}) \cap C^{0,\theta}([0,\tau], D((-\mathcal{A})^\beta)),$$

with $\theta = \min\{\beta, \frac{1}{2} - \beta\}$, there exist a time $T_\infty \in (0, +\infty]$ and a unique solution

$$u \in C([0, T_\infty), \mathcal{V}) \cap C^1((0, T_\infty), \mathcal{H})$$

of problem (DD).

Theorem [S. Nicaise and P., 2017]

Assume that conditions $(HD1), (HD2)$ as well as

$$f(0, y) = 0, \quad \forall\ y \in \mathbb{R},$$

are satisfied. With α_0 from the assumption $(HD1)$, we assume that

$$\alpha_0 < \varepsilon,$$

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$$\|u_0(t)\|_{\mathcal{V}} < \gamma_0 K, \quad \forall\ t \in [-\tau, 0],$$

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Delay diffusion equations

$$\|u(t)\|_{\mathcal{H}} \leq M e^{-\tilde{\omega} t} \quad \forall\; t \geq 0\,,$$

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Concrete examples

1. **Diffusive logistic equations with delay.**

$$f(x,y) = ax - bx^2 + cxy,$$

with $a, b, c \in L^\infty(\Omega)\,.$

$$\|u(t)\|_{\mathcal{H}} \le Me^{-\tilde{\omega}t} \quad \forall\ t \ge 0\,,$$

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1. **Diffusive logistic equations with delay.**

$$f(x,y) = ax - bx^2 + cxy,$$

with $a, b, c \in L^\infty(\Omega)$.
In such a case, the condition (HD1) holds with $\alpha_0 = \sup_\Omega |a + \varepsilon|$, and $n_1 = n_2 = 1$. Hence local existence holds for any $d \le 3$, while exponentiel decay for sufficiently small initial data holds under the additional assumption that

$$\sup_\Omega a < 0,$$

in the case of Neumann boundary conditions,

Delay diffusion equations

and

$$\sup_{\Omega} |a| < \lambda_1,$$

in the other cases.

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2. **The modified Hutchinson equation**.

In that case, f is given by (see [Memory (1991), Friesecke (1993)])

$$f(x,y) = \alpha x(1 + \beta y + \gamma y^2 + \delta y^3),$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. In such a case, the condition (HD1) holds with $\alpha_0 = |\alpha + \varepsilon|$, and $n_1 = n_2 = 3$. Hence local existence holds for any $d \leq 2$, while exponentiel decay for sufficiently small initial data holds under the additional assumption $\alpha < 0$ in the case of Neumann boundary conditions, and $-\lambda_1 < \alpha < \lambda_1$ in the other cases.

Delay diffusion equations

3. **A cubic nonlinearity**. The case where f is given by

$$f(x, y) = -x^2 y$$

considered in [Lighbourne and Rankin (1981), Friesecke (1993)] is also covered by our setting in the case of Dirichlet or Robin boundary conditions, since (HD1) holds with $\alpha_0 = 0$ and $n_1 = n_2 = 2$. Therefore local existence and exponentiel decay for sufficiently small initial data hold for any $d \leq 2$.

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• If f is globally Lipschitz, i.e., $P_1 = 0$ and P_2 is a positive constant, we can alternatively use the last abstract theorem and obtain exponential decay for small delays.

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... e auguri Piermarco!