Mayer Control Problem with Probabilistic Uncertainty on Initial Positions and Velocities

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An Optimal Control Problem

$$\dot{x}(t) = f(x(t), u(t)),\ u(t) \in U,\ t \in [0, T],$$

<u>**Goal:**</u> **Minimize** $J(x(\cdot)) = g(x(T))$ **over solutions s.t.** $x(0) = x_0$**,**
- The initial position x_0 **is not exactly known but only** μ_0 **a probabilistic description is available. (** $\forall A \subseteq \mathbb{R}^d$, $\mu_0(A)$ **is the probability that the initial position lies in the set** A**.)**
- At every point of $supp(\mu_0)$ **may correspond different controls- hence different velocities.**
- Possibility of "division of mass" from x_0 **can start different trajectories with total probability is equal to one.**

Dynamical system on $\mathcal{P}(\mathbb{R}^d)$

The conservation of mass along the trajectory $\boldsymbol{\mu} = \{\mu_t\}_{t\in[0,T]}$

$$\begin{cases} \partial_t \mu_t + \operatorname{div}(v_t \mu_t) = 0, \quad t \in [0,T] \\ \mu|_{t=0} = \mu_0, \\ v_t(x) \in F(x) := f(x,U), \qquad \qquad \textbf{for } \mu_t\textbf{-almost every } x \in \mathbb{R}^d, \end{cases}$$

Minimize $\mathcal{J}(\boldsymbol{\mu}) := \mathcal{G}(\mu_T) := \int_{\mathbb{R}^d} g(x)\, d\mu_T(x)$.

<u>Remark</u> If $v_t(\cdot)$ is Lipschitz then μ_t is the image measure of μ_0 by the flow at time t of the ODE $\dot{x}(t) = v_t(x(t))$.

Objectives

Study the corresponding value function

$$V(s,\mu) = \inf\left\{\mathcal{G}(\mu_T) : \{\mu_t\}_{t\in[s,T]} \in \mathcal{A}^F_{[s,T]}(\mu)\right\}.$$

- Regularity

- Dynamic Programming Principle

- Hamilton Jacobi characterization

- Extension to differential games

Bibliographical comments

- Continuity equation Ambrosio-Gigli-Savare
- Control case Cavagnari-Marigonda, Applications in Bressan-Zhang, Colombo-Lecureux-Mercier ....
- Hamilton Jacobi Equation on the Wasserstein space, Cardaliaguet-Quincampoix, Ambrosio-Gigli-Savare ....
- Differential games Jimenez-Quincampoix

Contents

Assumptions and Preliminaries

The trajectories $\boldsymbol{\mu} = \{\mu_t\}_{t\in[0,T]}$ can be represented as a superposition of trajectories defined in $[0,T]$ of a given differential inclusions

$$\dot{x}(t) \in F(x(t))$$

weighted by a probability measure μ on the initial state

($\boldsymbol{F}$) $F : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ **is a Lipschitz continuous set-valued map with nonempty compact convex values;**

($\boldsymbol{\mathcal{G}}$) $\mathcal{G} : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ **is bounded and Lipschitz continuous w.r.t. W_2 metric.**

Transport plan and Wasserstein distance

$\mathcal{P}_2(\mathbb{R}^d) = \{\mu \in \mathcal{P}(\mathbb{R}^d) : \mathrm{m}_2(\mathbb{R}^d) := \int_{\mathbb{R}^d} \|x\|^2 \, d\mu(x) < +\infty\}$

For $\mu_1, \mu_2 \in \mathcal{P}_2(\mathbb{R}^d)$. We define the set of *admissible transport plans* between μ_1 and μ_2 by setting

$$\Pi(\mu_1, \mu_2) = \{\gamma \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) : \pi_i \sharp \gamma = \mu_i, \; i = 1, 2\}.$$

$$W_2^2(\mu_1, \mu_2) = \inf_{\gamma \in \Pi(\mu_1, \mu_2)} \int_{\mathbb{R}^d \times \mathbb{R}^2} |x_1 - x_2|^2 \, d\gamma(x_1, x_2).$$

If $\mu_1, \mu_2 \in \mathcal{P}_2(\mathbb{R}^d)$ then the above infimum is actually a minimum, and we define $\Pi_o(\mu_1, \mu_2)$ the set of optimal γ.

Admissible trajectories

Let $a < b$**,** $\mu \in \mathcal{P}_2(\mathbb{R}^d)$**.** $\boldsymbol{\mu} = \{\mu_t\}_{t\in[a,b]} \subseteq \mathcal{P}_2(\mathbb{R}^d)$ **is an *admissible trajectory* starting from** μ **on** $[a,b]$ **if there exists a family of vector-valued measures** $\vec{\boldsymbol{\nu}} = \{\vec{\nu}_t\}_{t\in[a,b]} \subseteq \mathcal{M}(\mathbb{R}^d;\mathbb{R}^d)$ **such that**

- $\partial_t\mu_t + \operatorname{div}\vec{\nu}_t = 0$ **in the sense of distributions, and** $\mu_a = \mu$**,**
- $|\vec{\nu}_t| \ll \mu_t$ **for a.e.** $t \in [a,b]$**, i.e., the total variation** $|\vec{\nu}_t|$ **of** $\vec{\nu}_t$ **is absolutely continuous w.r.t.** μ_t **for a.e.** t **;**
- $\dfrac{\vec{\nu}_t}{\mu_t}(x) \in F(x)$ **for a.e.** $t \in [a,b]$ **and** μ_t**-a.e.** $x \in \mathbb{R}^d$**.**

(we will say that $\boldsymbol{\mu}$ **is *driven* by** $\vec{\boldsymbol{\nu}}$**)** $\mathcal{A}^F_{[a,b]}(\mu)$ **is the set of all admissible trajectories.**

Properties of Admissible Trajectories

- Superposition Principle: For any trajectory $\boldsymbol{\mu} = \{\mu_t\}_{t\in[0,T]} \in \mathcal{A}_T^F$ there exists a probability measure $\boldsymbol{\eta} \in \mathcal{P}(\mathbb{R}^d \times \Gamma_T)$, with $\Gamma_T = C^0([0,T];\mathbb{R}^d)$ endowed with the $\sup$ **norm such that**

(i) $\boldsymbol{\eta}$ **is concentrated on the pairs** (x,γ) **such that** γ **is a solution to**

$$\dot{\gamma}(t) \in F(\gamma(t)),\ \gamma(0) = x.$$

(ii) $\mu_t = e_t\sharp\boldsymbol{\eta}$ **for all** $t \in [0,T]$**,**

where $e_t(x,\gamma) := \gamma(t)$**.**

- **Closedness** $\mathcal{A}^F_{[0,T]}(\mu)$ **is closed for the distance** $d(\mu,\mu') := \sup_{t\in[0,T]} W_2(\mu_t,\mu'_t)$

- **Compactness If** $\mu^n \in \mathcal{A}^F_{[0,T]}(\mu_0^n)$ **and** $\sup_n m_2(\mu_0^n) < +\infty$ **then** $(\mu^n)_n$ **has a convergent subsequence for** d**.**

- **Concatenation**

- **Estimate If**] $\mu_t = e_t\sharp\boldsymbol{\eta}$ **for all** $t \in [a,b]$**, then for** $s_1, s_2 \in [a,b]$

$$\|e_{s_1} - e_{s_2}\|_{L^2_{\boldsymbol{\eta}}} \le Ce^{2(b-a)C}\left(1 + \min_{i=1,2} \mathrm{m}_2^{1/2}(\mu_{s_i})\right)|s_1 - s_2|,$$

Estimates for trajectories

Proposition 1 (Gronwall-like estimate in W_2) ***Assume that*** F ***satisfies*** $(\boldsymbol{F})$***. Let*** $a, b \in \mathbb{R}$ ***with*** $a < b$***. Then there exists*** $K > 0$ ***such that given*** $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$***,*** $\boldsymbol{\mu} = \{\mu_t\}_{t\in[a,b]} \in \mathcal{A}_{[a,b]}(\mu)$ ***it is possible to find*** $\boldsymbol{\nu} = \{\nu_t\}_{t\in[a,b]} \in A_{[a,b]}(\nu)$ ***satisfying***

$$W_2(\mu_t, \nu_t) \leq K \cdot W_2(\mu, \nu), \text{ \textbf{\textit{for all}} } t \in [a, b].$$

Existence of trajectories with prescribed initial velocity

Proposition 2 ***Let*** $a, b \in \mathbb{R}$***,*** $a < b$***,*** $\mu \in \mathcal{P}_2(\mathbb{R}^d)$***, Then for every*** $v_a \in L^2_\mu(\mathbb{R}^d)$ ***such that*** $v_a(x) \in F(x)$ ***for*** μ***-a.e.*** $x \in \mathbb{R}^d$ ***there exist*** $\boldsymbol{\eta} \in \mathcal{P}(\mathbb{R}^d \times \Gamma_{[a,b]})$ ***such that*** $\boldsymbol{\mu} = \{e_t \sharp \boldsymbol{\eta}\}_{t \in [a,b]} \in \mathcal{A}^F_{[a,b]}(\mu)$ ***and***

$$\lim_{t \to a^+} \int_{\mathbb{R}^d \times \Gamma_{[a,b]}} \langle \varphi \circ e_0(x, \gamma), \frac{e_t(x,\gamma) - e_a(x,\gamma)}{t - a} \rangle \, d\boldsymbol{\eta}(x, \gamma)$$

$$= \int_{\mathbb{R}^d} \langle \varphi(x), v_a(x) \rangle \, d\mu(x).$$

Regularity of the Value

The value function $V : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ **given by**

$$V(s,\mu) = \inf \left\{ \mathcal{G}(\mu_T) : \{\mu_t\}_{t\in[s,T]} \in \mathcal{A}^F_{[s,T]}(\mu) \right\}.$$

is bounded and for every $K \geq 0$**, it is Lipschitz continuous on the set**

$$\{(t,\mu) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^d),\ \mathrm{m}_2(\mu) \leq K\,\}$$

Dynamic Programming Principle

For all $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ **and** $\tau \in [0, T]$ **we have**

$$V(\tau, \mu) = \inf \left\{ V(s, \mu_s) : \{\mu_t\}_{t \in [\tau, T]} \in \mathcal{A}^F_{[\tau, T]}(\mu),\ s \in [\tau, T] \right\},$$

i.e., $V(\tau, \mu_\tau) \leq V(s, \mu_s)$ **for all** $\tau \leq s \leq T$ **and** $\{\mu_t\}_{t \in [\tau, T]} \in \mathcal{A}^F_{[\tau, T]}(\mu)$**, and** $V(\tau, \mu_\tau) = V(s, \mu_s)$ **for all** $\tau \leq s \leq T$ **if and only if** $\{\mu_t\}_{t \in [\tau, T]}$ **is an optimal trajectory for** μ**.**

Representation of Optimal Plan

Lemma 3 ***Let*** $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$***,*** $\gamma \in \Pi_o(\mu, \nu)$***. Then***

- $\exists! p^{\mu}_{\gamma} \in L^2_{\mu}(\mathbb{R}^d)$ ***and*** $q^{\nu}_{\gamma} \in L^2_{\nu}(\mathbb{R}^d)$ ***such that for all*** $\varphi \in L^2_{\mu}(\mathbb{R}^d, \mathbb{R}^d) \cap L^2_{\nu}(\mathbb{R}^d, \mathbb{R}^d)$ ***we have***

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \langle \varphi(x), x - y \rangle \, d\gamma(x, y) = \int_{\mathbb{R}^d} \langle \varphi, p^{\mu}_{\gamma} \rangle \, d\mu = \int_{\mathbb{R}^d} \langle \varphi, q^{\mu}_{\gamma} \rangle \, d\nu.$$

- ***we have*** $p^{\mu}_{\gamma} = \mathrm{Id}_{\mathbb{R}^d} - \mathrm{Bar}_1(\gamma)$***,*** $q^{\nu}_{\gamma} = \mathrm{Id}_{\mathbb{R}^d} - \mathrm{Bar}_1(\gamma^{-1})$ ***where***

$$\mathrm{Bar}_1(\gamma)(x_1) = \int_{\mathbb{R}^d} y \, d\gamma_{x_1}(y), \ \textit{\textbf{for } } \mu\textit{\textbf{-a.e.}} \ x_1 \in \mathbb{R}^d,$$

with $\gamma = \mu \otimes \gamma_{x_1}$***.***

On Viscosity Superdifferential

Let $w : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$**,** $(\bar{t}, \bar{\mu}) \in]0,T[\times \mathcal{P}_2(\mathbb{R}^d)$**,**
How to make a variation on μ **variable ?**
Naive Idea : $(I+\varphi)\sharp\bar{\mu}$ **for** $\varphi \in L^2_{\bar{\mu}}(\mathbb{R}^d)$

- **it works with** $\varphi(x) = f(x,u)$

But

- **in general** $\varphi(x) = f(x,u_x)$
- **division of mass** $\varphi(x)$ **is not "single valued" at** x

Viscosity δ-Superdifferential

Definition 4 ***Let*** $w : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$***,*** $(\bar t, \bar\mu) \in]0,T[\times\mathcal{P}_2(\mathbb{R}^d)$***,*** $\delta > 0$***.*** $(p_{\bar t}, p_{\bar\mu}) \in \mathbb{R} \times L^2_{\bar\mu}(\mathbb{R}^d)$ ***belongs to*** $D^+_\delta w(\bar t, \bar\mu)$ ***if***

i.) $\exists\ \bar\nu$***,*** $\gamma \in \Pi_o(\bar\mu, \bar\nu)$ $\forall \varphi \in L^2_\mu(\mathbb{R}^d, \mathbb{R}^d) \cap L^2_\nu(\mathbb{R}^d, \mathbb{R}^d)$ ***we have*** $\int_{\mathbb{R}^d\times\mathbb{R}^d} \langle \varphi(x), x - y\rangle \, d\gamma(x,y) = \int_{\mathbb{R}^d} \langle \varphi(x), p_{\bar\mu}(x)\rangle \, d\mu(x)$.

ii.) for all $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ ***we have***

$$w(t,\mu) - w(\bar t, \bar\mu) \le p_t(t - \bar t) + \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d} \langle x_2, x_3 - x_1\rangle \, d\tilde\mu(x_1, x_2, x_3) + \\ + \delta\sqrt{(t-\bar t)^2 + W^2_{2,\tilde\mu}(\bar\mu, \mu)} + o(|t - \bar t| + W_{2,\tilde\mu}(\bar\mu, \mu)),$$

$\forall \tilde\mu \in \mathcal{P}(\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d)$ ***s.t.*** $\pi_{12}\sharp\tilde\mu = (\mathrm{Id}_{\mathbb{R}^d}, p_{\bar\mu})\sharp\bar\mu$***,*** $\pi_{13}\sharp\tilde\mu \in \Pi(\bar\mu, \mu)$***.***

Viscosity δ-Subdifferential

$$D_\delta^- w(\bar{t}, \bar{\mu}) = D_\delta^+(-w)(\bar{t}, \bar{\mu})$$

Transport Multi plan

Let $\gamma \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$ **be a transport plan, and let** $\mu_3 \in \mathcal{P}_2(\mathbb{R}^d)$**.** **We set** $\mu_1 = \pi_1 \sharp \gamma$ **and**

$$\Pi(\gamma, \mu_3) := \{\tilde{\mu} \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d) : \pi_{12} \sharp \tilde{\mu} = \gamma,\ \pi_3 \sharp \tilde{\mu} = \mu_3\},$$
$$\Pi_o(\gamma, \mu_3) := \{\tilde{\mu} \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d) : \pi_{12} \sharp \tilde{\mu} = \gamma,\ \pi_{13} \sharp \tilde{\mu} \in \Pi_o(\mu_1, \mu_3)\}.$$

Given $\tilde{\mu} \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d)$**,** $i, j = 1, 2, 3$**, we set** $\mu_i = \pi_i \sharp \tilde{\mu}$ **and**

$$W^2_{2,\tilde{\mu}}(\mu_i, \mu_j) = \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d} |x_i - x_j|^2 \, d\tilde{\mu}(x_1, x_2, x_3).$$

Clearly, $W_{2,\tilde{\mu}}(\mu_i, \mu_j) \geq W_2(\mu_i, \mu_j)$ **for all** $i, j = 1, 2, 3$**.**

Lemma 5 ***Let*** $\gamma_{12}, \gamma_{13} \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ ***be such that*** $\pi_1 \sharp \gamma_{12} = \pi_1 \sharp \gamma_{13} = \mu_1 \in \mathcal{P}(\mathbb{R}^d)$***. Then there exists*** $\tilde{\mu} \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d)$ ***such that*** $\pi_{12} \sharp \tilde{\mu} = \gamma_{12}$ ***and*** $\pi_{13} \sharp \tilde{\mu} = \gamma_{13}$***. In particular, if*** $\gamma_{12} = \mu_1 \otimes \gamma_{12}^{x_1}$***,*** $\gamma_{13} = \mu_1 \otimes \gamma_{13}^{x_1}$***, and*** $\tilde{\mu} = \mu_1 \otimes \tilde{\mu}_{x_1}$***, we have*** $\tilde{\mu}_{x_1} \in \Pi(\gamma_{12}^{x_1}, \gamma_{13}^{x_1})$ ***for*** μ_1***-a.e.*** $x_1 \in \mathbb{R}^d$***. The measure*** $\tilde{\mu}$ ***is unique if*** γ_{12} ***or*** γ_{13} ***are induced by a transport map.***

Remarks on subdifferential

If ii.) of Definition to hold only for $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ ***induced by a transport map*** $\mu = (\mathrm{Id}+\varphi)\sharp\bar{\mu}$**, Then** $\exists! \tilde{\mu} \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d)$ **such that** $\pi_{12}\sharp\tilde{\mu} = (\mathrm{Id}, p_{\bar{\mu}})\sharp\bar{\mu}$ **and** $\pi_{13} = (\mathrm{Id}, \mathrm{Id}+\varphi)\sharp\bar{\mu}$**, and we have** $\tilde{\mu} = (\mathrm{Id}, p_{\bar{\mu}}, \mathrm{Id}+\varphi)\sharp\bar{\mu}$**. Then** $W_{2,\tilde{\mu}}(\bar{\mu}, \mu) = \|\varphi\|_{L^2_{\bar{\mu}}}$**, and we recover the same dδ-superdifferential of Cardaliaguet-Quincampoix (cf** $\bar{\mu} \ll \mathcal{L}^d$**).**

More generally, in item ii.) of Definition we consider absolutely continuous $\boldsymbol{\mu} = \{\mu_s\}_{s\in[0,t]}$ **curves** $\bar{\mu}$ **to** μ**, represented by** $\boldsymbol{\eta} \in \mathcal{P}(\mathbb{R}^d \times \Gamma_t)$ **satisfying** $\mu_s = e_s\sharp\boldsymbol{\eta}$**, then we can choose** $\tilde{\mu} \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d)$ **to be** $\tilde{\mu} = (e_0, p_{\bar{\mu}} \circ e_0, e_t)\sharp\boldsymbol{\eta}$**, recovering the same δ-superdifferential of Cavagnari-Marigonda-Nguyen**

Hamilton Jacobi Equation

$$\partial_t w(t,\mu) + \mathcal{H}(\mu, Dw(t,\mu)) = 0, \tag{1}$$

where $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ **and** $p \in L^2_\mu(\mathbb{R}^d)$**.** $w : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ **is**

- **a** ***subsolution*** **of** (1) **if** w **is u.s.c. and** $\exists C > 0$ **s.t.**

  $$p_t + \mathcal{H}(\mu, p_\mu) \geq -C\delta,$$

  for all $(t,\mu) \in]0,T[\times\mathcal{P}_2(\mathbb{R}^d)$**,** $(p_t, p_\mu) \in D^+_\delta w(t_0,\mu_0)$**, and** $\delta > 0$**.**

- **a** ***supersolution*** **of** (1) **if** w **is l.s.c. and** $\exists C > 0$ **s.t.**

  $$p_t + \mathcal{H}(\mu, p_\mu) \leq C\delta,$$

  for all $(t,\mu) \in]0,T[\times\mathcal{P}_2(\mathbb{R}^d)$**,** $(p_t, p_\mu) \in D^-_\delta w(t_0,\mu_0)$**, and** $\delta > 0$**.**

Comparison Theorem

Consider an Hamiltonian function $\mathcal{H}$ satisfying

- **positive homogeneity: for every $\lambda \geq 0$, $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, $p \in L^2_\mu(\mathbb{R}^d)$ we have $\mathcal{H}(\mu, \lambda p) = \lambda \mathcal{H}(\mu, p)$;**
- **dissipativity: $\exists k \geq 0$, $\forall \mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$, $\gamma \in \Pi_o(\mu, \nu)$, defined $p^\mu_\gamma = \mathrm{Id}_{\mathbb{R}^d} - \mathrm{Bar}_1(\gamma)$, $q^\nu_\gamma = \mathrm{Id}_{\mathbb{R}^d} - \mathrm{Bar}_1(\gamma^{-1})$, we have**

$$\mathcal{H}_F(\mu, p_\mu) - \mathcal{H}F(\nu, q_\nu) \leq k W_2^2(\mu, \nu).$$

Let w_1 and w_2 be a bounded and Lipschitz continuous sub and super solution respectively. Then

$$\inf_{(s,\mu) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^d)} w_2(s, \mu) - w_1(s, \mu) = \inf_{\mu \in \mathcal{P}_2(\mathbb{R}^d)} w_2(T, \mu) - w_1(T, \mu).$$

$$\mathcal{H}_F(\mu, p_\mu) :=$$

$$\inf \left\{ \int_{\mathbb{R}^d} \langle p_\mu(x), v_\mu(x) \rangle \, d\mu(x) : \begin{array}{l} v_\mu : \mathbb{R}^d \to \mathbb{R}^d \textbf{ Borel map} \\ v_\mu(x) \in F(x) \textbf{ for } \mu\textbf{-a.e. } x \in \mathbb{R}^d \end{array} \right\}.$$

Proposition **The Hamiltonian satisfies regularity properties need for the Comparison Theorem**

Main result

Theorem 6 ***Let*** $T > 0$***,*** $F : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ ***be a Lipschitz continuous set-valued map with nonempty compact convex values,*** $\mathcal{G} : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ ***be a bounded and Lipschitz continuous map. Then for any*** $K \geq 0$***, the value function*** $V(\cdot)$ ***is the unique Lipschitz continuous solution of the equation***

$$
\begin{cases} \partial_t w(t,\mu) + \mathcal{H}_F(\mu, Dw(t,\mu)) = 0, \\ w(T,\mu) = \mathcal{G}(\mu), \end{cases} \tag{2}
$$

stated on the set $\{(t,\mu) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^d),\ \mathrm{m}_2(\mu) \leq K\,\}$***.***

First player acts on the system

$$\partial_t \mu_t + \mathrm{div}(v_t \mu_t) = 0,\ v_t(x) \in F(x),\ \textbf{for } \mu_t \textbf{ a.e. } x \in \mathbb{R}^d\ t \in [0,T],$$

while the second player controls the system

$$\partial_t \nu_t + \mathrm{div}(\theta_t \nu_t) = 0,\ \theta_t(x) \in G(x),\ \textbf{for } \nu_t \textbf{ a.e. } x \in \mathbb{R}^d\ t \in [0,T].$$

Associated to both above dynamics, the following cost is defined

$$\mathcal{J} := \mathcal{G}(\mu_T, \nu_T),$$

that the first and the second player wish to minimize and maximize, respectively.

A *strategy* for the first player $\alpha : \mathcal{A}^G_{[t_0,T]} \to \mathcal{A}^F_{[t_0,T]}$. **is *nonanticipative with delay*** τ **if** $\exists \tau > 0$ **such that given** $t_0 \le s \le T$, $\boldsymbol{\nu}^i = \{\nu^i_t\}_{t\in[t_0,T]} \in \mathcal{A}^G_{[t_0,T]}$, $i = 1, 2$, **satisfying** $\nu^1_t = \nu^2_t$ **for all** $t_0 \le t \le s$, **and set** $\alpha(\boldsymbol{\nu}^i) = \{\mu^i_t\}_{t\in[t_0,T]}$, $i = 1, 2$, **we have** $\mu^1_t = \mu^2_t \; \forall t_0 \le t \le \min\{s + \tau, T\}$.

$\mathcal{A}(t_0, \mu_0)$ **is a set of strategies for the initial measure** μ_0

Lemma 7 (Normal form) ***Let*** $t_0 < \tau < T$. ***For any*** $(\alpha, \beta) \in \mathcal{A}_\tau(t_0) \times \mathcal{B}_\tau(t_0)$ ***there is a unique pair*** $(\boldsymbol{\mu}, \boldsymbol{\nu}) \in \mathcal{A}^F_{[t_0,b]} \times \mathcal{A}^G_{[t_0,b]}$ ***such that*** $\alpha(\boldsymbol{\nu}) = \boldsymbol{\mu}$ ***and*** $\beta(\boldsymbol{\mu}) = \boldsymbol{\nu}$.

Strategies and Values

$$J(t_0, \mu_0, \nu_0, \alpha, \beta) = \mathcal{G}\left(\mu_T, \nu_T\right),$$

$$V^+(t_0, \mu_0, \nu_0) = \inf_{\alpha \in \mathcal{A}(t_0,\mu_0)} \sup_{\beta \in \mathcal{B}(t_0,\nu_0)} J(t_0, \mu_0, \nu_0, \alpha, \beta)$$

$$V^-(t_0, \mu_0, \nu_0) = \sup_{\beta \in \mathcal{B}(t_0,\nu_0)} \inf_{\alpha \in \mathcal{A}(t_0,\mu_0)} J(t_0, \mu_0, \nu_0, \alpha, \beta).$$

Proposition $V^{\pm}(\cdot)$ **are bounded and locally Lipschitz continuous.**

A nonanticipative Lemma

$T > 0$, $t_0 \in [0,T]$, $\bar{\mu} \in \mathcal{P}_2(\mathbb{R}^d)$. $\exists \xi_{t_0}^{F,\bar{\mu}} : \mathcal{A}^F_{[t_0,T]} \to \mathcal{A}^F_{[t_0,T]}(\bar{\mu})$ $\exists K > 0$ **s.t. given** $\boldsymbol{\mu}^{(i)} = \{\mu_t^{(i)}\}_{t\in[0,T]} \in \mathcal{A}^F_{[t_0,T]}$, $i = 1,2$, **and set** $\boldsymbol{\mu}^{(3)} = \{\mu_t^{(3)}\}_{t\in[0,T]} = \xi_{t_0}^{F,\mu}(\boldsymbol{\mu}^{(1)})$, $\boldsymbol{\mu}^{(4)} = \{\mu_t^{(4)}\}_{t\in[0,T]} = \xi_{t_0}^{F,\bar{\mu}}(\boldsymbol{\mu}^{(2)})$,

i.) $W_2(\mu_t^{(1)}, \mu_t^{(3)}) \le K W_2(\mu_{t_0}^{(1)}, \mu_{t_0}^{(3)})$ **for all** $t \in [t_0, T]$;

ii.) if there exists $t_0 < s < T$ **such that** $\mu_t^{(2)} = \mu_t^{(1)}$ **for all** $t \in [t_0, s]$ **then** $\mu_t^{(4)} = \mu_t^{(3)}$ **for all** $t \in [t_0, s]$.

Notice that $\mu_{t_0}^{(3)} = \bar{\mu}$; **moreover,** $\forall \alpha \in \mathcal{A}_\tau(t_0)$, $\xi_{t_0}^{F,\bar{\mu}} \circ \alpha : \mathcal{A}^G_{[t_0,T]} \to \mathcal{A}^F_{[t_0,T]}(\mu)$ **is a nonanticipative strategy with delay** τ.

Dynamic Programming Principle

$$V^+(t_0, \mu^0, \nu^0) =$$

$$\inf_{\alpha \in \mathcal{A}(t_0, \mu^0)} \sup_{\beta \in \mathcal{B}(t_0, \nu^0)} \left\{ V^+(t_1, \mu_{t_1}, \nu_{t_1}) : \begin{array}{l} \boldsymbol{\mu} = \{\mu_t\}_{t \in [t_0, T]} = \alpha(\boldsymbol{\nu}) \\ \boldsymbol{\nu} = \{\nu_t\}_{t \in [t_0, T]} = \beta(\boldsymbol{\mu}) \end{array} \right\},$$

Hamiltonian

(3) $\mathcal{H}_{(}\mu, \nu, p_\mu, p_\nu) =$

$$\inf_{\substack{v(\cdot)\in L^2_\mu(\mathbb{R}^d) \\ v(x)\in F(x)\ \mu\textbf{-a.e.}x}} \int_{\mathbb{R}^d} \langle p_\mu(x), v(x)\rangle \, d\mu(x)$$

$$+ \sup_{\substack{w(\cdot)\in L^2_\nu(\mathbb{R}^d) \\ w(x)\in G(x)\ \nu\textbf{-a.e.}x}} \int_{\mathbb{R}^d} \langle p_\nu(x), w(x)\rangle \, d\nu(x).$$

Second Main Result

Theorem 8 (Existence of a value and its characterization) ***The game has a value, i.e.,*** $V^+ = V^- =: V$ ***and*** V ***is the unique Lipschitz continuous viscosity solution of the Hamilton-Jacobi-Bellman equation*** $\partial_t V + \mathcal{H}_{PE}(\mu, \nu, D_\mu V, D_\nu V) = 0$***,*** $V(T, \mu, \nu)$ $\mathcal{G}(\mu, \nu)$***.***

Extensions

- **Bolza Problem**
- **Cost with congestion**

$$\mathcal{J}(\mu) = \mathcal{G}(\mu_T) + \int_0^T L(\mu_t)dt$$

with

$L(\mu) = \int_{I\!R^d} h(x, \frac{\mu}{\lambda}(x))d\lambda$ **if** $\mu << \lambda$
and $L(\mu) = +\infty$ **else**

Thank you for your attention