Stability Analysis and Time-Discretization Schemes for Bang-Bang Optimal Control Problems

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joint work with V. M. Veliov, J. Preininger, and M. Quincampoix

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New trends in control theory and PDE's - On the occasion of the 60th birthday of Piermarco Cannarsa

Outline

The Problem and Some Motivations

The Linear-Quadratic Optimal Control Problem

- Let $x(\cdot; x_0, u) = x(\cdot) : [0,T] \to \mathbb{R}^n$ be the AC solution of

$$\begin{cases} \dot{x}(s) = A(s)x(s) + B(s)u(s) & s \in [0,T] \\ x(0) = x_0. \end{cases}$$

  - $U = [-1,1]^m$ is the control set,
  - $u : [0,T] \to U$ is any Lebesgue measurable function
- $A(s)$, $B(s)$, Q, q, $W(s)$, $S(s)$ and $T \in \mathbb{R}$ are given.
- Bolza's Problem: for $x_0 \in \mathbb{R}^n$, over all controls $u : [0,T] \to U$,

$$\min \underbrace{\frac{1}{2} x(T)^\top Q x(T) + q^\top x(T) + \int_0^T \left(\frac{1}{2} x^\top W x + x^\top S u \right) dt}_{J(x,u)}.$$

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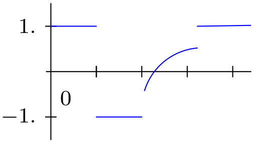
Linear-control systems with B-B solutions

- Affine-control systems in Milyutin-Osmolovskii (1998), Mauer-Osmolovskii (2003), Agrechev-Stefani-Zezza (2002), etc (sufficient conditions for optimality)
- Affine-control systems, Felgenhauer, Poggiolini, Stefani (2003-2015) (structural stability)
- Linear systems, Quinquampoix-Veliov (2013) and Seydenschwanz (2015), etc (stability analysis)
- Motivations: linear-control systems appear in several applications such as biology and medicine (see Ledzewick-Schättler), study of switched/hybrid systems, etc...



Minimum Principle and B-B Optimal Controls

Let $(\hat{x}, \hat{u})$ be an optimal solution. Then, there exists $\hat{p} \in W^{1,\infty}$ such that $(\hat{x}, \hat{u}, \hat{p})$ solves: for a.e. $t \in [0, T]$,

$$
\begin{aligned}
0 &= \dot{x}(t) - A(t)x(t) - B(t)u(t),\ x(0) = x_0, \\
0 &= \dot{p}(t) + A(t)^\top p(t) + W(t)x(t) + S(t)u(t), \\
0 &\in \underbrace{B(t)^\top p(t) + S(t)^\top x(t)}_{\sigma(t)} + N_U(u(t)), \\
0 &= p(T) - Qx(T) - q,
\end{aligned}
\tag{PMP}
$$

By defining $\hat{\sigma} = B^\top \hat{p} + S^\top \hat{x}$, for all $j = 1, \ldots, m$,

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\hat{u}_j(t) = \begin{cases} -\operatorname{sgn}(\hat{\sigma}_j(t)) & \text{if } \hat{\sigma}_j(t) \neq 0, \\ \text{undet}, & \text{if } \hat{\sigma}_j(t) = 0. \end{cases}
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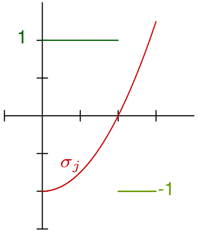

Stability Analysis under Perturbation

$F : X \rightrightarrows Y$, where $X := W^{1,1}_{x_0} \times W^{1,1} \times \mathcal{U}$, $Y := L^1 \times L^1 \times L^\infty \times \mathbb{R}^n$,

$$F(x,p,u) := \begin{pmatrix} \dot{x} - Ax - Bu \\ \dot{p} + A^\top p + Wx + Su \\ B^\top p + S^\top x + N_U(u) \\ p(T) - Qx(T) - q \end{pmatrix}, \quad \text{(PMP)} \Leftrightarrow 0 \in F(x,p,u).$$

F: solution mapping. Stability analysis: study of the continuity of the solutions of $y \in F(x,u,p)$ with respect to perturbation y.

Stability analysis and discretizations, some references

Dontchev-Hager (1993), Dontchev-Malanowski (1998), more recently Bonnans-Festa (2015), etc. They generally require

a smoothness of the optimal control

b strictly coercive cost functional/ strong second order optimality conditions

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Stability Analysis and Metric Regularity-type Properties

Bang-Bang Structure and Assumptions

Let $(\hat{x}, \hat{p}, \hat{u})$ be an optimal solution.

(R) The matrix-valued functions A, B, W, S have Lipschitz first derivatives; Q and $W(t)$ are symmetric $\forall\, t \in [0, T]$.

(C)

$$\frac{1}{2} z(T)^\top Q z(T) + \int_0^T \frac{1}{2} z^\top W z + z^\top S v \; dt \geq 0$$

for all $(z, v) \in \mathcal{F} \setminus \mathcal{F}$, $\mathcal{F}$ is the set of admissible processes.

(BB) There exist $\kappa \geq 1$ and $\alpha, \tau > 0$ such that for each $j \in \{1, \ldots, m\}$ and $s \in [0, T]$ with $\hat{\sigma}_j(s) = 0$ it holds that

$$|\hat{\sigma}_j(t)| \geq \alpha |t - s|^\kappa \quad \forall t \in [s - \tau, s + \tau] \cap [0, T].$$

κ = controllability index.

Strong Subregularity of the Mapping F

- $(x, p, u) \in X := W^{1,1}_{x_0} \times W^{1,1} \times \mathcal{U}, \mathcal{U} := \{u \in L^1 : u(t) \in [-1, 1]\}$,
- $y \in Y := L^1 \times L^1 \times L^\infty \times \mathbb{R}^n$.

Theorem (Preininger-S.-Veliov)

Let $(\hat{x}, \hat{p}, \hat{u})$ be a solution of PMP such that (BB) is fulfilled with index κ. Then for any $b > 0$ there exists $c > 0$ such that for any $y \in Y$ with $\|y\|_Y \leq b$, any $(x, p, u) \in X$ solving $y \in F(x, p, u)$ satisfies

$$\|x - \hat{x}\|_{1,1} + \|p - \hat{p}\|_{1,1} + \|u - \hat{u}\|_1 \leq c\|y\|_Y^{\frac{1}{\kappa}}.$$

- Hölder metric sub-regularity of $F : X \rightrightarrows Y$
- b can be any, and $c = c(b)$ depends in an explicit way on b (linear if $\kappa = 1$).
- Applications to the analysis of error estimates
- This property of F is not robust! $\Rightarrow$ ...

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Strongly Bi-Metric Regularity

Introducing $\widetilde{Y} := L^\infty \times L^\infty \times W^{1,\infty} \times \mathbb{R}^n$, endowed with $\tilde{d}_Y$.

Theorem ((Preininger-S.-Veliov))

Let $\hat{z} = (\hat{x}, \hat{p}, \hat{u})$ be a solution to the PMP such that (BB) is fulfilled with $\kappa = 1$ and suppose that $S^\top B$ is symmetric matrices-valued. Then there exist $\beta > 0$, $\zeta > 0$, and $a > 0$ such that

- *for any $y_1, y_2 \in B_{\tilde{d}_Y}(0; \beta)$, there exists unique z_1, $z_2 \in B_X(\hat{z}; a)$ such that $y_1 \in F(z_1)$ and $y_2 \in F(z_2)$,*
- *for such z_1 and z_2 it holds that $d_X(z_1, z_2) \le \zeta d_Y(y_1, y_2)$.*

- Recall $Y := L^1 \times L^1 \times L^\infty \times \mathbb{R}^n$ endowed with d_Y, so $\tilde{Y} \subset Y$.
- Lyusternik-Graves type theorems can be extended to bi-metric regular maps.
- If the norm were $W^{1,1}$ or L^∞ in u, sufficient conditions for the MR of F are known: Dontchev-Hager (1993), Dontchev-Malanowski (1998). They require essentially H_{uu} to be positive definite.

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A High-Order Time Discretization Scheme

- Goal: introducing a new time-discretization scheme of high-order of convergence and computing approximations of the optimal control with same bang-bang structure.
- Recall that for Runge-Kutta Methods (Hager-Dontchev-Veliov 2000, etc.) second-order optimality conditions and smoothness of the optimal control are required.

Idea: Volterra-Fliess Series

- u: any admissible control. x: solution of $\dot{x} = Ax + Bu$.
- Volterra-Fliess expansion. Given $N \in \mathbb{N}$, $h = T/N$, $t_i = ih$,

$$x(t_{i+1}) = [\underbrace{I + hA + \frac{h^2}{2}(A^2 + A')}_{A_i}]x(t_i) + \underbrace{(B + hAB)}_{B_i} \underbrace{\int_{t_i}^{t_{i+1}} u(s)ds}_{hz_1}$$

$$+ \underbrace{(-AB + B')}_{C_i} \underbrace{\int_{t_i}^{t_{i+1}} (s - t_i)u(s)ds}_{h^2 z_2} + O(h^3),$$

(all data evaluated at t_i).

- $$z_1 := \int_0^1 u(t)dt, \quad z_2 := \int_0^1 tu(t)dt.$$

References. Approximations in control theory using Volterra-Fliess expansions, Veliov (1989), Ferretti (1997).

Idea: Volterra-Fliess Series

By varying $u(\cdot)$ in the set of all admissible controls on $[0,1]$, the couple $(z_1, z_2) \in \mathbb{R}^{2m}$ generates the set $\mathbb{R}^{2m} \supset Z^m = \prod_1^m Z$, where

$$Z := \int_0^1 \begin{pmatrix} 1 \\ s \end{pmatrix} [-1,1] ds := \left\{ \int_0^1 \begin{pmatrix} 1 \\ s \end{pmatrix} f(s) ds : f \text{ selection of } [-1,1] \right\}.$$

Explicitly,

$$Z = \{(\alpha, \beta) : \alpha \in [-1,1], \beta \in (\phi_1(\alpha), \phi_2(\alpha))\},$$

where $\phi_{1,2}(\alpha) := \frac{1}{4}\left(\mp 1 + 2\alpha \pm \alpha^2\right)$.

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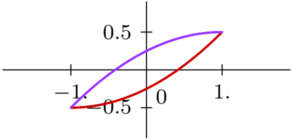

Finite-Dimensional Optimization Problem

Given $N \in \mathbb{N}$, $h = T/N$, $t_i := hi$,

$$\begin{array}{ll} \text{min} & \frac{1}{2} x_N^\top (Q+q) x_N + \frac{h}{2} \sum_{i=0}^{N-1} \text{linear-quadratic-affine form of } (u_i, v_i, x_i) \\ \text{subj to} & x_{i+1} = x_i + h(A_i x_i + B_i u_i + h C_i v_i) i = 0, \dots, N-1 \\ & x_0 \text{ given} \\ & (u_i, v_i) \in Z^m \quad i = 0, \dots, N-1. \end{array} \qquad (P^h)$$

where, for $i = 0, \dots, N-1$,

$$A_i := A(t_i) + \frac{h}{2}(A(t_i)^2 + A'(t_i)),$$
$$B_i := B(t_i) + hA(t_i)B(t_i), \quad C_i := -A(t_i)B(t_i) + B'(t_i).$$

Z^m is strictly convex – and bounded by quadratic curves in any control dimension.

Idea of the Proof of the rate of convergence

Combination of

1. Stability of the problem under perturbations

2. Consistency of the problem and its approximation

Let $\{(x_i, u_i, v_i, p_i)\}_{i=0}^{N-1}$ be the solution of the Karush-Kuhn-Tucker conditions of problem P^h.
We embed $\{(x_i, u_i, v_i, p_i)\}_{i=0}^{N-1} \hookrightarrow (x^N, p^N, u^N) \in W^{1,1} \times W^{1,1} \times L^1$ in such a way that the residual y^N, $(y^N \in F(x^N, p^N, u^N))$, satisfies $\| y^N \| \leq ch^2$. Thus,

$$\| x^N - \hat{x} \|_{1,1} + \| p^N - \hat{p} \|_{1,1} + \| u^N - \hat{u} \|_1 \underbrace{\leq}_{\text{sub-reg of } F} c \| y^N \|^{1/k} \leq \tilde{c}h^{2/k}.$$

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Main Result

Assumptions

- (C)- convexity and (R)- regularity of the data
- (BB)- pure bang-bang controls with controllability index κ
- (S)- symmetricity of the values of $S^\top B(\cdot)$

Theorem (S.-Veliov)

Let $(\hat{x}, \hat{p}, \hat{u})$ be the optimal triple for (P). Then $\forall N \in \mathbb{N}$ Problem (P^h) - $h = 1/N$ - has a solution $\{(x_i, u_i, v_i, p_i)\}$. Moreover, if the continuous embedding of (u_i, v_i) is u^N, it holds that

$$\max_{k=0,\dots,N} \left(|x_k - \hat{x}(t_k)| + |p_k - \hat{p}(t_k)| \right) + d^{\#}(u^N, \hat{u}) \leq c\, h^{2/\kappa}.$$

- $d^{\#}(u, \hat{u}) := meas(t \in [0, T] : \ u(t) \neq \hat{u}(t))$.
- If u is the result of an Euler method, then $d^{\#}(u, \hat{u}) \leq c\, h^{1/\kappa}$.
- For Runge-Kutta scheme method, the error is $O(h)$.

Numerical experiments

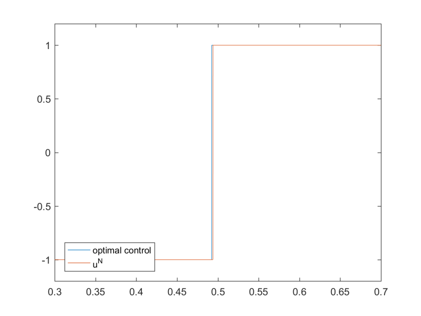
Example (Control of the double integrator)

min $\quad -0.1y(1) + \int_0^1 \frac{1}{2}(x(t))^2 dt$

subj to $\quad \dot{x} = y, \quad x(0) = 1,$

$\dot{y} = u, \quad y(0) = 0.1,$

$u \in [-1, 1].$



N	10	20	30	40	60
e_N	$1.50 \cdot 10^{-3}$	$3.64 \cdot 10^{-4}$	$1.54 \cdot 10^{-4}$	$9.35 \cdot 10^{-5}$	$3.97 \cdot 10^{-5}$
e_N/h^2	0.150	0.146	0.139	0.150	0.143

Conclusions

Some themes of today

- Stability analysis for some LQ problems with bang-bang controls
- High-order time-discretizations for discontinuous optimal controls

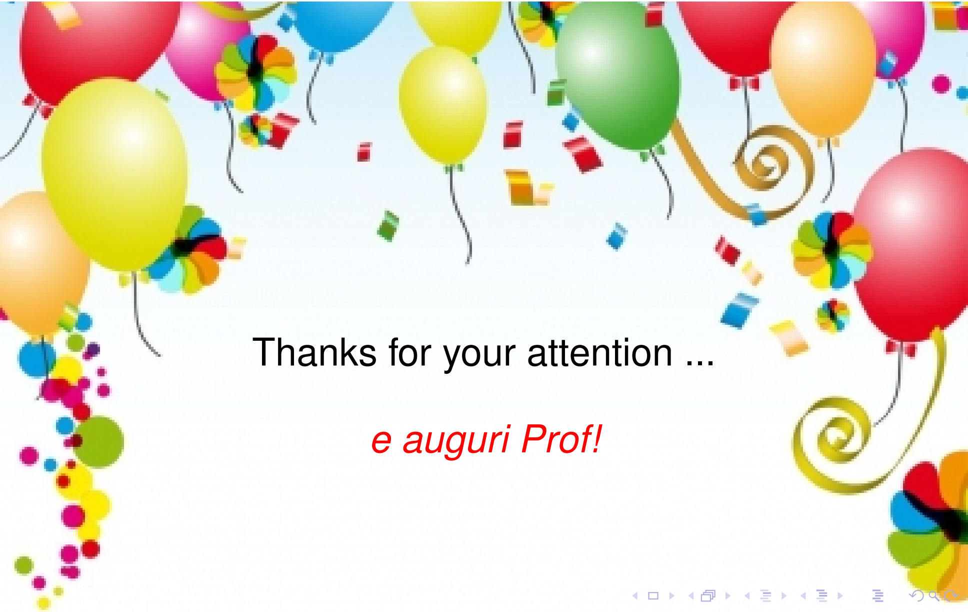
Present and future work

Extension to the case of

- concatenations of singular and bang-bang arcs, and
- control-affine systems $\dot{x} = f(x,t) + B(x,t)u(t)$

The numerical schemes is capable; the stability analysis is not understood yet!

Some related works: Poggiolini-Stefani (sufficient opt conditions and structural stability) and Felgenhauer (time-discretization) (2003-15), Aronna-Bonnans-al (second order opt conditions) (2012-16).



Thanks for your attention ...

e auguri Prof!