## Martingale Optimal Transport

H. Mete Soner

ETH Zürich
and
Swiss Finance Institute

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## Introduction

Problem

Context

## Martingale Optimal Transport

Forward-Start options
VIX Option

## Robust Duality

Model
Outline of duality proof

- Yan Dolinsky, Hebrew University,
- Ibrahim Ekren, ETH Zürich,
- Matti Kiiski, ETH Zürich,
- Matteo Burzoni, ETH Zürich,
- Frank Riedel, University of Bielefeld.


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For a given Radon measures $\mu, \nu$ on $\mathbb{R}^{d}$, the Martingale Optimal Transport on $\Omega=\mathbb{R}^{d} \times \mathbb{R}^{d}$ is

$$
\mathcal{D}(\xi):=\sup _{\mathbb{Q} \in \mathbb{M}(\mu, \nu)} \mathbb{E}_{\mathbb{Q}}[\xi], \quad \xi \in \mathbb{L}^{0}(\Omega)
$$

where a probability measure $\mathbb{Q}$ on $\Omega$ is in $\mathbb{M}(\mu, \nu)$ if and only if it has marginals $\mu$ and $\nu$, i.e.,

$$
\begin{aligned}
& \mathbb{Q}(\{(x, y) \in \Omega: x \in A\})=\mu(A) \\
& \mathbb{Q}(\{(x, y) \in \Omega: y \in A\})=\nu(A)
\end{aligned}
$$

for every Borel $A \subset \mathbb{R}^{d}$ and $\mathbb{Q}$ is a martingale, i.e.,

$$
\mathbb{E}_{\mathbb{Q}}[\gamma(x) \cdot(y-x)]=0
$$

for every bounded $\gamma$.

Let $0 \leq t_{1}<\ldots<t_{n} \leq T$ and $\mu_{1}, \ldots, \mu_{n}$ be given Radon measures on $\mathbb{R}^{d}$. The Martingale Optimal Transport on $\Omega=C\left([0,1] ; \mathbb{R}^{d}\right)$ with the usual filtration is

$$
\mathcal{D}(\xi):=\sup _{\mathbb{Q} \in \mathbb{M}_{\mu_{1}, \ldots, \mu_{n}}} \mathbb{E}_{\mathbb{Q}}[\xi], \quad \xi \in \mathbb{L}^{0}(\Omega)
$$

where a probability measure $\mathbb{Q} \in \mathbb{M}_{\mu_{1}, \ldots, \mu_{n}}$ if and only if $\mathbb{Q}$ restricted to time $t_{k}$ has distribution $\mu_{k}$ for all $k=1, \ldots, n$, ie.,

$$
\mathbb{Q}\left(\left\{\omega \in \Omega: \omega_{t_{k}} \in A\right\}\right)=\mu_{k}(A),
$$

for every Bored $A \subset \mathbb{R}^{d}$ and the canonical process $\mathbb{S}_{t}(\omega):=\omega_{t}$ is a $\mathbb{Q}$ martingale.

There are many interesting mathematical questions:

- Duality and Attainment.
- Financial meaning and Fundamental Theorem of Asset Pricing.
- When is $\mathbb{M}_{\mu_{1}, \ldots, \mu_{n}}$ non-empty? we know by a result of Strassen that $\mathbb{M}(\mu, \nu)$ is nonempty if and only if $\mu$ and $\nu$ are in convex order, ie.,

$$
\mu(\varphi) \leq \nu(\varphi) \quad \text { for all convex } \varphi
$$

- More restrictions on $\mathbb{Q}$ 's.


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- Proper use of mathematics in social sciences is challenging, new, quite different and difficult.
- The common problem can be summarised as decision making under uncertainty.
- Indeed, everybody is constantly involved in some sort of decision making with imperfect information. We may imagine that these decisions are taken to optimise certain outcomes.
- What is not clear and is very different from standard engineering applications are :
no clear objective,
no certain nature given dynamics,
no law of large number, or one realisation of the randomness.

We would like to analyze an uncertain outcome $\xi: \Omega \rightarrow \mathbb{R}^{d}$.

- In risk management develop a risk measure

$$
\rho: \mathbb{L}^{0}(\Omega) \rightarrow \mathbb{R}
$$

Then, $\rho(\xi)$ can be used to determine the cash reserves needed for this risk position. It allows the managers to distribute or aggregate the risk among the subdivisions.

- In pricing, we also want a linear pricing operator

$$
V: \mathbb{L}^{0}(\Omega) \rightarrow \mathbb{R}
$$

But here one does not have to be as conservative as above.

- In hedging, one would like to reduce the risk associated to $\xi$ by appropriate trading in liquid assets.

There are economic factors that determine the outcome $\xi$.

- If we believe that there is a stationary stochastic model $\mathbb{P}$ governing these processes and assume that it is slowly varying in time, then we can estimate it well. (There is large historical data). $\mathbb{P}$ is called the physical measure.
- Even if there is a historical measure $\mathbb{P}$, a linear pricing rule has to take into account the riskiness of the position $\xi$. This results

$$
V(\xi)=\mathbb{E}_{\mathbb{Q}}[\xi] \neq \mathbb{E}_{\mathbb{P}}[\xi]
$$

where $\mathbb{Q}$ is the risk neutral (or martingale) measure or pricing operator.

From financial data we have two types of information,

- Information about the historical performance under the physical measure $\mathbb{P}$;
- Estimation of future densities under the risk neutral measure $\mathbb{Q}$ - in particular the use of the options data.

Today emphasis is on the second.

In current financial markets Call and Put options are highly liquid. Their future random pay-offs are given by,

$$
c_{K}\left(\mathbb{S}_{T}\right):=\left(\mathbb{S}_{T}-K\right)^{+} \quad \text { Call, } \quad p_{K}\left(\mathbb{S}_{T}\right):=\left(K-\mathbb{S}_{T}\right)^{+} \quad \text { Put },
$$

where $\mathbb{S}_{T}$ is the future, random stock price and $K \geq 0$ is a pre-agreed strike.

According to theory, their prices are given through a risk neutral measure $\mathbb{Q}$. Here $\mathbb{Q}$ is a measure on the whole path of the stock price process. Let $\nu$ be the distribution of $\mathbb{S}_{T}$ under $\mathbb{Q}$. Let $C(K)$ be the price of Call and $P(K)$ be the price of Put. Then,

$$
C(K)=\nu\left(c_{K}\right)=\int(s-K)^{+} \nu(d s), \quad P(K)=\nu\left(p_{K}\right)
$$

Now suppose that we know $C(K)$ 's or $P(K)$ 's for all $K \geq 0$. And as I will argue later, this is not very unrealistic assumption.

Since the set of functions $\left\{c_{K}: k \geq 0\right\}$ or $\left\{p_{K}: k \geq 0\right\}$ form a separating class, above information uniquely determines the future pricing distribution $\nu$.

So by observing prices of Calls, Puts and maybe other options we obtain information about the future price distributions or the perception of the market participants about these prices.

Kantorovic problem is

$$
\sup _{\mathbb{Q} \in \mathcal{M}(\mu, \nu)} \mathbb{E}_{\mathbb{Q}}[\xi]:=\int_{\mathbb{R}^{d} \times \mathbb{R}^{d}} \xi(x, y) \mathbb{Q}(d x, d y)
$$

$\mathbb{Q} \in \mathcal{M}(\nu, \mu)$ iff it has marginals $\mu$ and $\nu$ (no martingale condition). This is a linear program and its convex dual is

$$
\inf \{\mu(g)+\nu(h): g(x)+h(y) \geq \xi(x, y), \forall(x, y)\}
$$

Let $\mathbb{Q} \in \mathcal{M}(\nu, \mu)$. Suppose $g, h$ satisfy the dual inequality

$$
g(x)+g(y) \geq \xi(x, y), \quad \forall(x, y)
$$

Integrate both sides with respect to $\mathbb{Q}$. Then,

$$
\mu(g)+\nu(h) \geq \int \xi(x, y) \mathbb{Q}(d x, d y)=\mathbb{E}_{\mathbb{Q}}[\xi]
$$

This proves the easy inequality in the dual.

Consider a financial market with

- trading is only allowed in two future time points, $0<t_{1}<t_{2}$;
- we can dynamically trade the stock whose future values will be $\mathbb{S}_{1}, \mathbb{S}_{2}$ and no assumption on them is made but $\mathbb{S}_{0}$ is given;
- we can also now buy any option of the form $g\left(\mathbb{S}_{1}\right)$ and $h\left(\mathbb{S}_{2}\right)$ for a known price of $\mu(g)$, respectively $\nu(h)$. Here $g, h$ are arbitrary and can be chosen by the investor. But probability measures $\mu$ and $\nu$, i.e. the linear pricing rules, are given.
- we want to super-replicate a claim $\xi\left(\mathbb{S}_{1}, \mathbb{S}_{2}\right)$. An example is $\xi\left(\mathbb{S}_{1}, \mathbb{S}_{2}\right)=\left(\mathbb{S}_{2}-\mathbb{S}_{1}\right)^{+}$, this is the forward-start Call option.

Mathematically, we want to minimise (with zero interest rate) the prices of the options $g\left(\mathbb{S}_{1}\right)$ and $h\left(\mathbb{S}_{2}\right)$, i.e.,

$$
\operatorname{minimize} \mu(g)+\nu(h)
$$

over all options $(g, h)$ that together with a dynamic trading strategy $\theta\left(\mathbb{S}_{1}\right)$, dominate the path-dependent claim $\xi\left(\mathbb{S}_{1}, \mathbb{S}_{2}\right)$. $\theta_{1}\left(\mathbb{S}_{1}\right)$ is the number shares will be bought at time $t_{1}$, and this decision will depend on the value of the stock at time $t_{1}$.

More compactly,

$$
\text { minimize } \mu(g)+\nu(h)
$$

over all $(g, h)$ so that there exists $\theta_{1}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$,

$$
\theta_{1}\left(\mathbb{S}_{1}\right) \cdot\left(\mathbb{S}_{2}-\mathbb{S}_{1}\right)+g\left(\mathbb{S}_{1}\right)+h\left(\mathbb{S}_{2}\right) \geq \xi\left(\mathbb{S}_{1}, \mathbb{S}_{2}\right)
$$

for all $\mathbb{S}_{1}, \mathbb{S}_{2} \in \mathbb{R}^{d}$.
Compared to optimal transport (Kantorovic dual) the red term is new. And the dual elements, i.e., the martingale measures, have to annihilate them.

The convex dual of the financial problem is the Martingale Optimal Transport:

$$
\operatorname{maximize} \mathbb{E}_{\mathbb{Q}}\left[\xi\left(\mathbb{S}_{1}, \mathbb{S}_{0}\right)\right]
$$

over all martingale measures $\mathbb{Q} \in \mathbb{M}(\nu, \mu)$, i.e, for $\Omega:=\mathbb{R}^{2}$, the canonical process

$$
\mathbb{S}_{0}\left(\left(\omega_{1}, \omega_{2}\right)\right)=\mathbb{S}_{0}, \quad \mathbb{S}_{1}\left(\left(\omega_{1}, \omega_{2}\right)\right)=\omega_{1}, \quad \mathbb{S}_{2}\left(\left(\omega_{1}, \omega_{2}\right)\right)=\omega_{2}
$$

with the canonical filtration is a $\mathbb{Q}$ martingale, i.e.,

$$
\mathbb{E}\left[\mathbb{S}_{2} \mid \mathbb{S}_{1}\right]=\mathbb{S}_{1}
$$

Assumptions are $\int y \mu(d y)=\int x \nu(d x)=\mathbb{S}_{0}$ and that they are in convex order. So that the set of martingale measures satisfying the constraint is non-empty by Strassen.

Suppose that for some $g, h$,

$$
g\left(\mathbb{S}_{1}\right)+h\left(\mathbb{S}_{2}\right)+\theta_{1}\left(\mathbb{S}_{1}\right) \cdot\left(\mathbb{S}_{2}-\mathbb{S}_{1}\right) \geq \xi\left(\mathbb{S}_{1}, \mathbb{S}_{2}\right)
$$

for some $\theta_{1}$, and for all $\left(\mathbb{S}_{1}, \mathbb{S}_{2}\right)$. Let $\mathbb{Q} \in \mathbb{M}(\nu, \mu)$ be a martingale measure. Then,

$$
\mathbb{E}^{\mathbb{Q}}\left[\theta_{1}\left(\mathbb{S}_{1}\right) \cdot\left(\mathbb{S}_{2}-\mathbb{S}_{1}\right)\right]=0
$$

and the marginals imply that

$$
\mathbb{E}^{\mathbb{Q}}\left[g\left(\mathbb{S}_{1}\right)\right]=\mu(g), \quad \mathbb{E}^{\mathbb{Q}}\left[g\left(\mathbb{S}_{2}\right)\right]=\nu(h)
$$

So if we integrate the first inequality with respect to $\mathbb{Q}$,

$$
\mu(g)+\nu(h) \geq \mathbb{E}_{\mathbb{Q}}\left[\xi\left(\mathbb{S}_{1}, \mathbb{S}_{0}\right)\right]
$$

VIX is a measure of expected volatility calculated as 100 times the square root of the expected 30-day variance of the $S \& P$ 500. The definition of realized variance over $N$ trading days is

$$
R V:=\frac{252}{N-1} \times \sum_{i=1}^{N-1} r_{i}^{2}
$$

where $r_{i}=\ln \left(\mathbb{S}_{i+1} / \mathbb{S}_{i}\right)$ is daily return. If the stock price process $\mathbb{S}_{t}$ is a diffusion of the form $d \mathbb{S}_{u}=\mathbb{S}_{u}\left[\mu_{u} d u+\sigma_{u} d B_{u}\right]$ where the processes $\mu, \sigma$ are arbitrary and $B$ is Brownian motion, then,

$$
R V=\frac{252}{N-1} \times \sum_{i=1}^{N-1} r_{i}^{2} \approx \frac{1}{T} \int_{0}^{T} \sigma_{u}^{2} d u
$$

Since

$$
d\left[\ln \left(\mathbb{S}_{u}\right)\right]=\frac{d \mathbb{S}_{u}}{\mathbb{S}_{u}}-\frac{\sigma_{u}^{2}}{2} d u
$$

For any martingale measure $\mathbb{Q}$, the first term is a martingale :

$$
\mathbb{E}_{\mathbb{Q}}[R V] \approx \frac{1}{T} \mathbb{E}_{\mathbb{Q}}\left(\int_{0}^{T} \sigma_{u}^{2} d u\right)=-\frac{2}{T} \mathbb{E}_{\mathbb{Q}}\left[\ln \left(\mathbb{S}_{T} / S_{0}\right)\right]
$$

Let $\nu$ be the distribution of $\mathbb{S}_{T}$ under $\mathbb{Q}$. Then

$$
V I X^{2}=\mathbb{E}_{\mathbb{Q}}[R V] \approx-\frac{2}{T}\left[\nu(\ln (\cdot))-\ln \left(\mathbb{S}_{0}\right)\right]
$$

Then, the question is how to determine $\nu$. Note we cannot determine $\mathbb{Q}$ and in general there are many of them.

Breeden \& Litzenberger '78 gives a formula for $\mathbb{E}_{\mathbb{Q}}\left[\ln \left(\mathbb{S}_{T}\right)\right]$ in terms of the Call, $C(k)$, and Put, $P(k)$, prices. We start with the identity (in fact, In is not important),

$$
\ln (x / a)=\frac{x-a}{a}-\int_{0}^{a} \frac{(k-x)^{+}}{k^{2}} d k-\int_{a}^{\infty} \frac{(x-k)^{+}}{k^{2}} d k
$$

Hence, with $x=\mathbb{S}_{T}$ and $a=\mathbb{S}_{0}$,

$$
\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[R V] & \approx-\frac{2}{T} \mathbb{E}_{\mathbb{Q}}\left[\ln \left(\mathbb{S}_{T} / \mathbb{S}_{0}\right)\right] \\
& =\frac{2}{T}\left[\int_{0}^{\mathbb{S}_{0}} \frac{P(k)}{k^{2}} d k+\int_{\mathbb{S}_{0}}^{\infty} \frac{C(k)}{k^{2}} d k\right] .
\end{aligned}
$$

Summarizing,

$$
\begin{aligned}
V I X^{2} & =\mathbb{E}_{\mathbb{Q}}[R V] \approx \frac{1}{T} \mathbb{E}_{\mathbb{Q}}\left(\int_{0}^{T} \sigma_{u}^{2} d u\right)=-\frac{2}{T} \mathbb{E}_{\mathbb{Q}}\left[\ln \left(\mathbb{S}_{T} / \mathbb{S}_{0}\right)\right] \\
& =\frac{2}{T}\left[\int_{0}^{\mathbb{S}_{0}} \frac{P(k)}{k^{2}} d k+\int_{\mathbb{S}_{0}}^{\infty} \frac{C(k)}{k^{2}} d k\right] \\
& =\frac{2}{T} \int_{0}^{\infty} \ln \left(x / \mathbb{S}_{0}\right) \nu(d x),
\end{aligned}
$$

where $\nu$ is density of $\mathbb{S}_{T}$ under any risk neutral measure and it is completely determined by the Call and Put prices.

In this example, we get an exact formula for VIX. This is the formula used by CBOE (Chicago Board of Exchange).


The powerful and flexible trading and risk management tool from the Chicago Board Options Exchange

## THE CBOE VOLATILITY INDEX - VIX

## The CBOE Volatlity Index ${ }^{\circledR}$ - VIX ${ }^{\circledR}$

In 1993, the Chicago Board Options Exchange ${ }^{\circledR}\left(\mathrm{CBOE}^{\circledR}\right)$ introduced the CBOE Volatility Index ${ }^{\circledR}$, VIX ${ }^{\circledR}$, which was originally designed to measure the market's expectation of 30 day volatility implied by at-the-money S\&P $100^{\circledR}$ Index $\left(\mathrm{OEX}^{\circledR}\right)$ option prices. VIX soon became the premier benchmark for U.S. stock market volatility. It is regularly featured in the Wall Street Journal, Barron's and other leading financial publications, as well as business news shows on CNBC, Bloomberg TV and CNN/Money, where VIX is often referred to as the "fear index."

Ten years later in 2003, CBOE together with Goldman Sachs, updated the VIX to reflect a new way to measure expected volatility, one that continues to be widely used by financial theorists, risk managers and volatility traders alike. The new VIX is based on the S\&P $500^{\circledR}$ Index (SPX ${ }^{\text {SM }}$ ), the core index for U.S. equities, and estimates expected volatility by averaging the weighted prices of SPX puts and calls over a wide range of strike prices. By supplying a script for replicating volatility exposure with a portfolio of SPX options, this new methodology transformed VIX from an abstract concept into a practical standard for trading and hedging volatility.

## Volatility as a Tradable Asset - VIX Futures \& Options

On March 24, 2004, CBOE introduced the first exchange-traded VIX futures contract on its new, all-electronic CBOE Futures Exchange ${ }^{\mathrm{SM}}\left(\mathrm{CFE}^{\circledR}\right)$. Two years later in February 2006, CBOE launched VIX options, the most successful new product in Exchange history. In less than five years, the combined trading activity in VIX options and futures has grown to more than 100,000 contracts per day.

The negative correlation of volatility to stock market returns is well documented and suggests a diversification benefit to including volatility in an investment portfolio. VIX futures and options are designed to deliver pure volatility exposure in a single, efficient package. CBOE/CFE provides a continuous, liquid and transparent market for VIX products that are available to all investors from the smallest retail trader to the largest institutional money managers and hedge funds.

## Beyond VIX

In addition to VIX, CBOE calculates several other volatility indexes including the CBOE Nasdaq- $100^{\circledR}$ Volatility Index (VXN ${ }^{\text {SM }}$ ), CBOE DJIA ${ }^{\circledR}$ Volatility Index (VXD ${ }^{\text {SM }}$ ), CBOE Russell $2000^{\circledR}$ Volatility Index ( $\mathrm{RVX}^{\mathrm{SM}}$ ) and CBOE S\&P $500^{\circledR}$ 3-Month Volatility Index (VXV ${ }^{\mathrm{SM}}$ ). Currently, VXD and RVX futures are listed on CFE; RVX options trade on CBOE.

In 2008, CBOE pioneered the use of the VIX methodology to estimate expected volatility of certain commodities and foreign currencies. The CBOE Crude Oil Volatility Index $\left(\mathrm{OVX}^{\mathrm{SM}}\right)$, CBOE Gold Volatility Index $\left(\mathrm{GVZ}^{\mathrm{SM}}\right)$ and CBOE EuroCurrency Volatility Index

## The VIX Calculation Step-by-Step

Stock indexes, such as the S\&P 500, are calculated using the prices of their component stocks. Each index employs rules that govern the selection of component securities and a formula to calculate index values.

VIX is a volatility index comprised of options rather than stocks, with the price of each option reflecting the market's expectation of future volatility. Like conventional indexes, VIX employs rules for selecting component options and a formula to calculate index values.

The generalized formula used in the VIX calculation ${ }^{\S}$ is:

WHERE...

$$
\begin{equation*}
\boldsymbol{\sigma}^{2}=\frac{2}{T}\left(\sum_{i} \frac{\Delta K_{i}}{K_{i}^{2}} e^{R T} Q\left(K_{i}\right)-\frac{1}{T}\left[\frac{F}{K_{0}}-1\right]^{2}\right. \tag{1}
\end{equation*}
$$

$\sigma$ is

$$
V I X / 100 \Rightarrow \mathrm{VIX}=\sigma \times 100
$$

T Time to expiration
F Forward index level derived from index option prices
$\mathrm{K}_{0} \quad$ First strike below the forward index level, F
$\mathrm{K}_{i} \quad$ Strike price of $i^{\text {th }}$ out-of-the-money option; a call if $\mathrm{K}_{i}>\mathrm{K}_{0}$ and a put if $\mathrm{K}_{i}<\mathrm{K}_{0}$; both put and call if $\mathrm{K}_{i}=\mathrm{K}_{0}$.
$\Delta \mathrm{K}_{\mathrm{i}} \quad$ Interval between strike prices - half the difference between the strike on either side of $\mathrm{K}_{\mathrm{i}}$ :

$$
\Delta \mathrm{K}_{\mathrm{i}}=\frac{K_{i+1}-K_{i-1}}{2}
$$

(Note: $\Delta \mathrm{K}$ for the lowest strike is simply the difference between the lowest strike and the next higher strike. Likewise, $\Delta \mathrm{K}$ for the highest strike is the difference between the highest strike and the next lower strike.)
$\mathrm{R} \quad$ Risk-free interest rate to expiration
$\mathrm{Q}\left(\mathrm{K}_{\mathrm{i}}\right) \quad$ The midpoint of the bid-ask spread for each option with strike $\mathrm{K}_{\mathrm{i}}$.
§ Please see "More than you ever wanted to know about volatility swaps" by Kresimir Demeterfi, Emanuel Derman, Michael Kamal and Joseph Zou, Goldman Sachs Quantitative Strategies Research Notes, March 1999.

| 710 | 209.90 | 214.90 | 1.10 | 2.40 | 835 | 115.60 | 120.80 | 30.30 | 35.40 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 720 | 200.00 | 205.70 | 1.10 | 3.00 | 840 | 111.90 | 117.10 | 31.80 | 36.90 |
| 725 | 195.20 | 200.90 | 2.00 | 3.00 | 845 | 108.20 | 113.40 | 32.90 | 38.00 |
| 730 | 190.60 | 195.60 | 1.40 | 3.50 | 850 | 107.00 | 109.70 | 34.10 | 39.60 |
| 740 | 180.80 | 186.50 | 1.80 | 4.00 | 855 | 101.30 | 106.30 | 35.90 | 39.00 |
| 750 | 171.30 | 177.00 | 3.20 | 3.90 | 860 | 97.40 | 103.00 | 37.40 | 42.50 |
| 760 | 162.10 | 167.10 | 2.85 | 5.00 | 865 | 94.80 | 99.50 | 38.60 | 43.70 |
| 770 | 152.80 | 157.80 | 4.00 | 5.60 | 870 | 91.30 | 96.00 | 40.00 | 45.10 |
| 775 | 148.20 | 153.20 | 3.90 | 5.90 | 875 | 87.90 | 92.60 | 41.70 | 47.10 |
| 780 | 143.60 | 148.60 | 4.10 | 6.70 | 880 | 83.60 | 88.90 | 43.30 | 48.70 |
| 790 | 134.60 | 140.10 | 4.90 | 7.50 | 885 | 80.30 | 85.80 | 44.80 | 50.30 |
| 800 | 125.60 | 131.10 | 6.10 | 7.50 | 890 | 77.50 | 83.00 | 47.00 | 52.10 |
| 805 | 121.00 | 126.00 | 6.40 | 9.10 | 895 | 74.00 | 79.50 | 48.60 | 53.80 |
| 810 | 116.50 | 121.50 | 7.10 | 9.50 | 900 | 70.80 | 76.40 | 50.20 | 55.40 |
| 815 | 112.20 | 117.20 | 7.50 | 10.30 | 905 | 67.60 | 73.10 | 52.20 | 57.20 |
| 820 | 107.60 | 113.20 | 8.20 | 10.80 | 910 | 64.60 | 70.10 | 54.00 | 59.50 |
| 825 | 103.70 | 108.90 | 8.90 | 11.50 | 915 | 62.40 | 67.10 | 56.30 | 61.50 |
| 830 | 99.10 | 104.30 | 9.00 | 13.00 | 920 | 59.10 | 64.00 | 57.80 | 63.30 |
| 835 | 95.30 | 100.30 | 9.80 | 13.70 | 925 | 56.20 | 61.70 | 60.30 | 65.80 |
| 840 | 90.90 | 95.90 | 10.60 | 14.30 | 930 | 53.00 | 58.50 | 62.90 | 67.90 |
| 845 | 86.70 | 91.90 | 11.50 | 15.20 | 935 | 50.30 | 55.80 | 64.60 | 70.10 |
| 850 | 82.60 | 88.00 | 13.50 | 16.00 | 940 | 47.60 | 52.70 | 67.00 | 72.60 |
| 855 | 78.70 | 83.90 | 13.50 | 17.50 | 945 | 45.70 | 50.40 | 69.50 | 75.00 |
| 860 | 74.90 | 79.90 | 14.50 | 18.70 | 950 | 44.80 | 47.70 | 72.20 | 76.60 |
| 865 | 70.90 | 76.10 | 15.50 | 19.90 | 955 | 40.40 | 45.30 | 74.70 | 79.70 |
| 870 | 67.20 | 72.30 | 17.10 | 20.90 | 960 | 38.10 | 43.20 | 77.30 | 82.30 |
| 875 | 63.80 | 68.60 | 18.20 | 22.20 | 965 | 35.60 | 40.70 | 79.80 | 85.30 |
| 880 | 59.80 | 65.30 | 19.20 | 24.00 | 970 | 33.40 | 38.90 | 82.60 | 87.90 |
| 885 | 56.60 | 61.60 | 20.30 | 25.40 | 975 | 32.40 | 36.10 | 85.70 | 90.70 |
| 890 | 53.70 | 58.40 | 22.10 | 26.80 | 980 | 28.90 | 34.00 | 88.60 | 93.40 |
| 895 | 50.30 | 55.00 | 23.90 | 29.00 | 985 | 27.00 | 32.00 | 91.60 | 96.80 |
| 900 | 46.20 | 51.70 | 25.50 | 29.00 | 990 | 25.40 | 30.30 | 94.70 | 99.80 |
| 905 | 43.40 | 48.90 | 27.20 | 32.30 | 995 | 23.20 | 28.60 | 97.80 | 103.00 |
| 910 | 40.00 | 45.10 | 29.20 | 34.20 | 1000 | 23.00 | 26.40 | 101.00 | 106.20 |
| 915 | 37.30 | 42.80 | 30.80 | 36.30 | 1005 | 20.00 | 24.70 | 104.30 | 109.50 |
| 920 | 35.20 | 39.10 | 35.20 | 38.10 | 1010 | 18.40 | 23.30 | 107.70 | 112.90 |
| 925 | 31.40 | 35.20 | 35.10 | 40.30 | 1015 | 17.10 | 21.20 | 111.10 | 116.30 |
| 930 | 31.00 | 33.90 | 37.40 | 42.90 | 1020 | 18.00 | 19.80 | 114.60 | 119.80 |
| 935 | 26.00 | 31.50 | 40.30 | 45.10 | 1025 | 14.40 | 18.20 | 118.20 | 123.40 |
| 940 | 26.00 | 29.00 | 42.50 | 48.10 | 1030 | 13.10 | 16.90 | 121.90 | 127.10 |
| 945 | 21.60 | 26.40 | 45.30 | 50.60 | 1035 | 11.80 | 15.80 | 125.60 | 130.60 |
| 950 | 21.60 | 24.40 | 47.60 | 53.10 | 1040 | 10.70 | 14.70 | 128.80 | 134.30 |
| 955 | 17.20 | 22.50 | 51.30 | 56.50 | 1045 | 9.30 | 13.60 | 133.10 | 138.60 |
| 960 | 15.50 | 19.60 | 54.40 | 59.40 | 1050 | 10.80 | 12.00 | 137.30 | 142.30 |
| 965 | 13.90 | 18.00 | 57.50 | 62.50 | 1055 | 8.10 | 10.60 | 140.90 | 146.60 |
| 970 | 12.00 | 16.40 | 60.80 | 66.00 | 1060 | 8.20 | 9.80 | 145.20 | 150.70 |
| 975 | 12.50 | 14.80 | 64.30 | 69.50 | 1065 | 6.50 | 9.00 | 148.90 | 154.40 |
| 980 | 9.30 | 13.10 | 67.80 | 73.00 | 1070 | 5.60 | 8.10 | 153.40 | 158.60 |
| 985 | 8.00 | 11.90 | 71.50 | 76.70 | 1075 | 5.10 | 7.60 | 157.40 | 162.90 |
| 990 | 7.40 | 9.90 | 75.10 | 80.50 | 1080 | 4.40 | 6.80 | 162.40 | 167.40 |
| 995 | 6.40 | 8.70 | 79.20 | 84.20 | 1085 | 3.80 | 6.20 | 166.40 | 172.10 |

CBOE Proprietary Information

| 1000 | 6.50 | 7.50 | 82.50 | 88.00 | 1090 | 3.30 | 5.50 | 171.30 | 176.30 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1005 | 4.40 | 6.80 | 86.90 | 92.50 | 1095 | 2.90 | 4.90 | 175.80 | 180.80 |
| 1010 | 3.60 | 6.00 | 91.10 | 96.50 | 1100 | 3.30 | 4.50 | 180.10 | 185.60 |
| 1015 | 3.10 | 5.10 | 95.40 | 101.10 | 1105 | 2.00 | 4.10 | 184.20 | 189.70 |
| 1020 | 3.50 | 4.60 | 99.80 | 105.40 | 1110 | 2.00 | 3.60 | 189.30 | 194.80 |
| 1025 | 2.00 | 4.10 | 104.70 | 109.70 | 1115 | 1.65 | 3.30 | 194.20 | 199.20 |
| 1030 | 1.60 | 3.50 | 108.50 | 114.00 | 1120 | 1.45 | 3.00 | 198.90 | 203.90 |
| 1035 | 1.20 | 3.10 | 113.60 | 119.10 | 1125 | 2.00 | 2.40 | 202.90 | 208.40 |
| 1040 | 1.25 | 2.50 | 118.50 | 123.50 | 1130 | 0.90 | 2.20 | 208.40 | 213.40 |
| 1045 | 0.85 | 2.10 | 123.20 | 128.20 | 1135 | 0.70 | 1.90 | 212.70 | 218.40 |
| 1050 | 1.30 | 1.80 | 127.70 | 133.20 | 1140 | 1.00 | 1.75 | 218.00 | 223.00 |
| 1055 | 0.40 | 1.65 | 132.50 | 138.00 | 1145 | 0.50 | 1.50 | 222.80 | 227.80 |
| 1060 | 0.90 | 1.40 | 137.60 | 142.60 | 1150 | 0.35 | 1.30 | 227.40 | 232.90 |
| 1065 | 0.20 | 1.25 | 142.20 | 147.70 | 1155 | 0.25 | 1.20 | 232.30 | 237.80 |
| 1070 | 0.15 | 1.15 | 147.30 | 152.30 | 1160 | 0.10 | 1.10 | 236.70 | 242.20 |
| 1075 | 0.50 | 1.00 | 152.20 | 157.20 | 1165 | 0.00 | 1.00 | 241.60 | 247.10 |
| 1080 | 0.40 | 0.95 | 156.90 | 162.40 | 1170 | 0.00 | 0.90 | 247.00 | 252.50 |
| 1085 | 0.20 | 0.90 | 161.80 | 167.30 | 1175 | 0.30 | 0.85 | 251.90 | 257.40 |
| 1090 | 0.30 | 0.85 | 167.00 | 172.00 | 1180 | 0.00 | 0.85 | 257.10 | 262.10 |
| 1095 | 0.20 | 0.75 | 171.70 | 177.20 | 1185 | 0.00 | 0.80 | 261.30 | 266.50 |
| 1100 | 0.30 | 0.45 | 176.90 | 181.90 | 1190 | 0.00 | 0.50 | 267.00 | 272.00 |
| 1105 | 0.20 | 0.75 | 181.60 | 187.10 | 1190 | 0.00 | 0.00 | 0.00 | 0.00 |
| 1110 | 0.20 | 0.35 | 186.80 | 191.80 | 1195 | 0.00 | 0.75 | 271.20 | 276.40 |
| 1115 | 0.05 | 0.80 | 191.80 | 196.80 | 1200 | 0.30 | 0.60 | 276.90 | 281.90 |
| 1120 | 0.05 | 0.50 | 196.80 | 201.80 | 1205 | 0.00 | 0.75 | 281.60 | 287.10 |
| 1125 | 0.15 | 0.50 | 201.50 | 207.00 | 1210 | 0.00 | 0.60 | 286.80 | 291.80 |
| 1130 | 0.05 | 0.90 | 206.70 | 211.70 | 1215 | 0.00 | 0.85 | 291.80 | 296.80 |
| 1135 | 0.05 | 0.90 | 211.70 | 216.70 | 1220 | 0.00 | 0.75 | 296.80 | 301.80 |
| 1140 | 0.05 | 0.70 | 216.70 | 221.70 | 1225 | 0.00 | 0.80 | 301.50 | 307.00 |
| 1145 | 0.05 | 0.95 | 221.70 | 226.70 | 1230 | 0.00 | 0.80 | 306.70 | 311.70 |
| 1150 | 0.05 | 0.35 | 226.70 | 231.70 | 1235 | 0.00 | 0.80 | 311.70 | 316.70 |
| 1155 | 0.05 | 0.95 | 231.70 | 236.70 | 1240 | 0.10 | 0.75 | 316.70 | 321.70 |
| 1160 | 0.05 | 0.50 | 236.70 | 241.70 | 1245 | 0.00 | 0.75 | 321.70 | 326.70 |
| 1165 | 0.05 | 0.35 | 241.70 | 246.70 | 1250 | 0.00 | 1.00 | 326.70 | 331.20 |
| 1170 | 0.05 | 0.45 | 246.40 | 251.90 | 1255 | 0.00 | 0.75 | 331.40 | 336.90 |
| 1175 | 0.05 | 0.15 | 251.40 | 256.90 | 1260 | 0.00 | 0.70 | 335.90 | 341.10 |
| 1180 | 0.05 | 0.80 | 256.40 | 261.90 | 1265 | 0.00 | 0.70 | 341.60 | 346.60 |
| 1185 | 0.05 | 0.25 | 260.90 | 266.10 | 1270 | 0.00 | 0.70 | 346.60 | 351.60 |
| 1190 | 0.05 | 0.50 | 266.60 | 271.60 | 1275 | 0.05 | 0.20 | 351.60 | 356.60 |
| 1195 | 0.05 | 1.00 | 271.60 | 276.60 | 1280 | 0.00 | 0.75 | 356.60 | 361.60 |
| 1200 | 0.05 | 0.15 | 276.60 | 281.60 | 1290 | 0.00 | 0.75 | 366.60 | 371.60 |
| 1205 | 0.05 | 1.00 | 281.60 | 286.60 | 1300 | 0.05 | 0.45 | 375.70 | 381.00 |
| 1210 | 0.05 | 0.50 | 286.60 | 291.60 | 1315 | 0.00 | 0.50 | 390.70 | 395.90 |
| 1215 | 0.05 | 0.50 | 291.60 | 296.60 | 1320 | 0.00 | 0.75 | 396.50 | 401.50 |
| 1220 | 0.05 | 1.00 | 296.60 | 301.60 | 1325 | 0.00 | 0.50 | 399.90 | 405.90 |
| 1225 | 0.00 | 1.00 | 301.60 | 306.60 | 1335 | 0.00 | 0.75 | 411.50 | 416.50 |
| 1230 | 0.00 | 1.00 | 306.60 | 311.60 | 1340 | 0.00 | 0.75 | 416.50 | 421.50 |
| 1235 | 0.00 | 0.75 | 311.60 | 316.60 | 1345 | 0.00 | 0.75 | 421.50 | 426.50 |
| 1240 | 0.00 | 0.50 | 316.60 | 321.60 | 1350 | 0.00 | 0.50 | 425.60 | 430.80 |
| 1245 | 0.00 | 0.15 | 321.60 | 326.60 | 1360 | 0.00 | 0.75 | 435.70 | 440.90 |
| 1250 | 0.05 | 0.10 | 326.60 | 331.60 | 1375 | 0.00 | 0.55 | 451.40 | 456.40 |

CBOE Proprietary Information

## Problem

Context

## Forward-Start options

VIX Option

## Robust Duality

Model
Outline of duality proof

Suppose that all right-continuous processes with left limits (namely càdlàg processes) are possible stock price processes, i.e.,

$$
\Omega:=\mathbb{D}:=\left\{\mathbb{S}:[0, T] \rightarrow \mathbb{R}_{+}^{d} \mid \mathbb{S}_{0}=(1, \ldots, 1) \text {, càdlàg }\right\} .
$$

All European options $g\left(\mathbb{S}_{T}\right)$ maturing only at the final time are traded. Its price is assumed to be $\nu(g)$. We assume that

$$
\int_{\mathbb{R}^{+}} x_{k} \nu(d x)=\mathbb{S}_{0}^{(k)}=1, \quad \forall k=1, \ldots, d
$$

This allows for martingale measures.

A general European path-dependent claim, $\xi$, is given,

$$
\xi=G(\mathbb{S}), \quad \text { where } G: \mathbb{D} \rightarrow \mathbb{R}
$$

The minimal super-replication cost also provides the upper bound of the price interval and is given by

$$
\begin{aligned}
& V(G):=\inf \left\{\int g d \nu \mid \exists \gamma\right. \text { admissible such that } \\
&\left.\int_{[0, T]} \gamma_{t}(\mathbb{S}) d \mathbb{S}_{t}+g\left(\mathbb{S}_{T}\right) \geq G(\mathbb{S}), \forall \mathbb{S} \in \mathbb{D}\right\}
\end{aligned}
$$

Clearly, admissibility restrictions on $\gamma$ are needed, such as a lower bound, predictability.

Dual refers to a probabilistic structure, which we now introduce. $\Omega:=\mathbb{D}$ is as before and let $\mathbb{S}$ be the canonical process and $\mathcal{F}_{t}$ be the canonical filtration.

As before $\mathbb{M}_{\nu}$ is the set of all martingale measures such that the probability distribution of $\mathbb{S}_{T}$ under $\mathbb{Q}$ is $\nu$, i.e., for every Borel subset $B \subset \mathbb{R}^{d}$,

$$
\mathbb{Q}\left(\mathbb{S}_{T} \in B\right)=\nu(B)
$$

## Theorem (DS : continuous (2012) PTRF - càdlàg (2013))

Assume that $G$ is bounded and uniformly continuous with respect to the Skorokhod metric. Then,

$$
V(G)=\sup _{\mathbb{Q} \in \mathbb{M}_{\nu}} \mathbb{E}_{\mathbb{Q}}[G(\mathbb{S})]
$$

Notice that this result is in analogy with the incomplete market result. Previous results fix probability measure $\mathbb{P}$ and consider only the martingale measures that are absolutely continuous with respect to $\mathbb{P}$. In the robust case, we fix the marginal $\nu$ but otherwise have no dominating measure.

## Problem

Context

## Forward-Start options

VIX Option

## Robust Duality

Model
Outline of duality proof

- Construct an approximating problem with countably many possible stock price process. This requires to approximate not only the possible stock values but also the jump times.
- In the countable market, probabilistic and robust hedging are same as we put non-zero probability to each process.
- We consider a super-replication problem $V^{(n)}$ with bounded portfolio positions (bound disappears in the limit).
- We use the constrained duality result of Föllmer \& Kramkov and a min-max theorem. The result is

$$
V^{(n)} \leq \sup _{\mathbb{Q} \in \mathbb{M}_{\nu}^{n}} \mathbb{E}^{\mathbb{Q}}[G]
$$

where $\mathbb{M}_{\nu}^{n}$ is a set of approximate martingale measures.

We want to prove two facts, with vanishing error e(n)

$$
\begin{aligned}
& \text { 1. } \quad V(G) \leq V^{(n)}+e(n) \\
& \text { 2. } \sup _{\mathbb{Q} \in \mathbb{M}_{\nu}^{n}} \mathbb{E}^{\mathbb{Q}}[G] \leq \sup _{\mathbb{Q} \in \mathbb{M}_{\nu}} \mathbb{E}^{\mathbb{Q}}[G]+e(n) .
\end{aligned}
$$

First is done by lifting : given a portfolio that super-replicates in the discrete (an easier) market, we lift it to a
super-replicating one in the original market. Main point is the price is this lifted portfolio is the same as the original one plus a uniformly controlled small error, $e(n)$.

Construction of the lift is technical due to measurability requirements.

What is left to prove is the following,

$$
\sup _{\mathbb{Q} \in \mathbb{M}_{\nu}^{n}} \mathbb{E}^{\mathbb{Q}}[G] \leq \sup _{\mathbb{Q} \in \mathbb{M}_{\nu}} \mathbb{E}^{\mathbb{Q}}[G]+e(n)
$$

Any element $\mathbb{Q} \in \mathbb{M}_{\nu}^{n}$ is an approximate martingale and satisfies the marginal constrained approximately.

Difficulty is the non-compactness of these spaces. Idea is for every $\mathbb{Q} \in \mathbb{M}_{\nu}^{n}$, we construct $\mathbb{Q}_{n} \in \mathbb{M}_{\nu}$ so that

$$
\mathbb{E}^{\mathbb{Q}}[G] \leq \mathbb{E}_{\mathbb{Q}_{n}}[G]+e(n)
$$

with a uniformly controlled error e(n).
We achieve this using semi-martingale decomposition, Doob's martingale inequality and a tool from weak convergence.

- Dupire 94, Local volatility and Dupire equation;
- Hobson 98, explicit hedges via Skorokhod embedding;
- Beiglböck, Henry-Labordère and Penkner, 2011, discrete time, duality and connection to optimal transport;
- Galichon, Henry-Labordère and Touzi, 2011, first result and formulation in continuous time and quasi-sure approach;
- Dolinsky and Soner, 2012, 2013, 2014, continuous time duality and discrete time market with transaction costs;
- Bouchard and Nutz, 2013, discrete time quasi-sure approach, fundamental paper studying the quasi-sure set-up;

FTAP= Fundamental Theorem of Asset Pricing.
MOT= Martingale Optimal Transport
SEP= Skorokhod Embedding Problem.

- Acciaio, Beiglböck and Schachermayer, 2013, FTAP in discrete time;
- Cox, Davis, Dobson, Huesmann, Klimmek, Obloj, 1998-2015; connections between SEP and MOT;
- Beiglböck, Cox and Huesmann, 2014, systematic construction of solutions to the SEP through MOT;
- Beiglböck, Friz and Sturm, 2011, local volatility counter example;
- Henry-Labordère, Obloj, Spoida, and Touzi, 2012, explicit solutions to the max-max;
- Acciaio, Beiglböck, Penkner, Schachermayer and Temme, 2013, a proof of Doob's inequality. Similar construction also appears in Burkholder 1992;
- Beiglböck and Siorpaes, 2012, A new proof of the Biechtelier theorem;
- Riedel, 2012, decision under uncertainty in one-step setting;
- Burzonni, Fritelli and Maggis, 2014, arbitrage in multi-step.
- Soner, Touzi, Zhang 2012, 2013, quasi-sure hedging.

Several papers are available and they are joint with

## Yan Dolinsky from Hebrew University

- Continuous path duality, P.T.R.F., 2014 ;
- Discrete time with frictions, Finance \& Stochastics, 2014 ;
- càdlàg paths, SPA; 2016.


## NICE YILLARA PIERMARCO

