Observability inequalities on measurable sets for the Stokes system and applications

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Stokes system

Here: $T > 0$ and $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded domain with a regular boundary $\partial\Omega$.

The Stokes system

$$\left| \begin{array}{lll} \mathbf{z}_t - \Delta \mathbf{z} + \nabla q = \mathbf{0} & \text{in} & Q = \Omega \times (0,T), \\ \operatorname{div} \mathbf{z} = 0 & \text{in} & Q = \Omega \times (0,T), \\ \mathbf{z} = \mathbf{0} & \text{on} & \Sigma = \partial\Omega \times (0,T), \\ \mathbf{z}(\cdot, 0) = \mathbf{z}_0 & \text{in} & \Omega. \end{array} \right.$$

The usual spaces in the context of fluid mechanics:

$$\mathbf{H} = \{\mathbf{y} \in \mathbf{L}^2(\Omega)^N; \ \operatorname{div} \mathbf{y} = 0, \ \mathbf{y} \cdot \boldsymbol{\nu} = 0 \text{ on } \partial\Omega\}$$

and

$$\mathbf{V} = \{\mathbf{y} \in \mathbf{H}_0^1(\Omega)^N; \ \operatorname{div} \mathbf{y} = 0\}.$$

The Stokes operator $\mathbf{A} : D(\mathbf{A}) \longrightarrow \mathbf{H}$ is defined by $\mathbf{A} = -P\Delta$, where $D(\mathbf{A}) = \big\{\mathbf{y} \in \mathbf{V} : \ \mathbf{A}\mathbf{y} \in \mathbf{H}\big\} = \mathbf{H}^2(\Omega) \cap \mathbf{V}$ and $P : \mathbf{L}^2(\Omega) = \mathbf{H} \oplus \mathbf{H}^\perp \longrightarrow \mathbf{H}$ is the Leray projection.

Some previous results

L^2-observability inequality for the Stokes system:

For any **open subset** $\omega \subset \Omega$*, there exists a positive constant* $C = C(N, \Omega, \mathcal{M}, T)$ *such that the observability estimate*

$$\|\mathbf{z}(T,\cdot)\|_{\mathbf{H}} \leq C \left(\iint_{\omega \times (0,T)} |\mathbf{z}(\mathbf{x},t)|^2 \, d\mathbf{x}dt \right)^{1/2} \quad \forall \mathbf{z}_0 \in \mathbf{H}$$

holds.

Remark: L^2-observability inequality $\Rightarrow$ L^2-null controllability.

- A. V. Fursikov, O. Yu. Imanuvilov. *Controllability of Evolution Equations*. Lecture Notes Series 34, Research Institute of Mathematics, Seoul National University, Seoul, 1996.
- F. W. Chaves-Silva, G. Lebeau. *Spectral inequality and optimal cost of controllability for the Stokes system*. ESAIM: COCV, 22 (2016) 1137-1162.

Observability inequality on measurable sets

L^1-observability inequality for Stokes system:

Theorem (Chaves-Silva, D. A. Souza and C. Zhang)

For any measurable subset $\mathcal{M} \subset \Omega \times (0, T)$ *with positive measure, i.e.* $|\mathcal{M}| > 0$, *there exists a positive constant* $C = C(N, \Omega, \mathcal{M}, T)$ *such that the observability estimate*

$$\|\mathbf{z}(T, \cdot)\|_{\mathbf{H}} \leq C \iint_{\mathcal{M}} |\mathbf{z}(\mathbf{x}, t)| \, d\mathbf{x}dt \quad \forall \mathbf{z}_0 \in \mathbf{H}$$

holds.

Consequences:

- $\mathbf{L}^\infty$- null controllability on measurable sets of positive measure;
- Bang-bang property for the time optimal control problem.

Strategy

- The propagation of smallness for real-analytic functions on measurable sets [1];
- Spectral inequality for Stokes system [2];
- Telescoping series method[3];

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[1] S. Vessella. *A continuous dependence result in the analytic continuation problem.* Forum Math., 11 (1999), 695–703.

[2] F. W. Chaves-Silva, G. Lebeau. *Spectral inequality and optimal cost of controllability for the Stokes system*. ESAIM: COCV, 22 (2016) 1137-1162.

[3] Apraiz, L. Escauriaza, G. Wang, C. Zhang. *Observability inequalities and measurable sets*. J. Eur. Math. Soc., 16 (2014), 2433–2475.

Spectral inequality on open sets

Let $\{\mathbf{e}_j\}_{j\geq 1}$ be the sequence of eigenfunctions of the Stokes system

$$\left| \begin{array}{lll} -\Delta \mathbf{e}_j + \nabla p_j = \lambda_j \mathbf{e}_j & \text{in} & \Omega, \\ \operatorname{div} \mathbf{e}_j = 0 & \text{in} & \Omega, \\ \mathbf{e}_j = \mathbf{0} & \text{on} & \partial\Omega, \end{array} \right.$$

with the eigenvalues $\{\lambda_j\}_{j\geq 1}$ satisfying $0 < \lambda_1 \leq \lambda_2 \leq \dots$ and $\lim\limits_{j\to\infty} \lambda_j = +\infty$.

The following inequality holds:

Theorem (Chaves-Silva and Lebeau '16)

$\forall \mathcal{O} \subset \Omega$ non-empty open set, there exists a constant $C = C(N, \Omega, \mathcal{O}) > 0$ such that

$$\sum_{\lambda_j \leq \Lambda} a_j^2 = \int_\Omega \left| \sum_{\lambda_j \leq \Lambda} a_j \mathbf{e}_j(\mathbf{x}) \right|^2 d\mathbf{x} \leq C e^{C\sqrt{\Lambda}} \int_{\mathcal{O}} \left| \sum_{\lambda_j \leq \Lambda} a_j \mathbf{e}_j(\mathbf{x}) \right|^2 d\mathbf{x},$$

for any sequence of real numbers $\{a_j\}_{j\geq 1} \in \ell^2$ and any positive number Λ.

Spectral inequality on measurable sets

Our second main result is an extension of the spectral inequality from open sets to measurable sets of positive measure:

Theorem (Chaves-Silva, D. A. Souza and C. Zhang)

Let $\omega \subset \Omega$ *be a measurable set with positive measure. Then, there exists a constant* $C = C(N, \Omega, |\omega|) > 0$ *such that*

$$\left(\sum_{\lambda_j \le \Lambda} a_j^2\right)^{1/2} = \left(\int_\Omega \left|\sum_{\lambda_j \le \Lambda} a_j \mathbf{e}_j(\mathbf{x})\right|^2 d\mathbf{x}\right)^{1/2} \le Ce^{C\sqrt{\Lambda}} \int_\omega \left|\sum_{\lambda_j \le \Lambda} a_j \mathbf{e}_j(\mathbf{x})\right| d\mathbf{x},$$

for all $\Lambda > 0$ *and any sequence of real numbers* $\{a_j\}_{j\ge 1} \in \ell^2$.

Two main difficulties:

- presence of the pressure;
- the equation satisfied by the "*curl*" is an equation without pressure but with no boundary conditions.

Idea of the proof

A key point is the following inequality:

Lemma (estimate of the propagation of smallness for real-analytic functions[4]*)*
Assume that $\mathbf{f} : B_{2R}(\mathbf{x}_0) \subset \mathbb{R}^N \longrightarrow \mathbb{R}^N$ *is real-analytic and verifies*

$$|\partial_x^\alpha \mathbf{f}(\mathbf{x})| \leq \frac{M|\alpha|!}{(\rho R)^{|\alpha|}}, \textit{ for } \mathbf{x} \in B_{2R}(\mathbf{x}_0),\ \alpha \in \mathbb{N}^N,$$

for some $M > 0$ *and* $0 < \rho \leq 1$.
For any measurable set $\omega \subset B_R(\mathbf{x}_0)$ *with positive measure, there are positive constants* $C = C(R, N, \rho, |\omega|)$ *and* $\theta = \theta(R, N, \rho, |\omega|)$, *with* $\theta \in (0,1)$, *such that*

$$\|\mathbf{f}\|_{\mathbf{L}^\infty(B_R(\mathbf{x}_0))} \leq C \left(\int_\omega |\mathbf{f}(\mathbf{x})|\, d\mathbf{x} \right)^\theta M^{1-\theta}.$$

Remark: we need to quantify the interior real-analytic estimates!

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[4] S. Vessella. *A continuous dependence result in the analytic continuation problem.* Forum Math., 11 (1999), 695–703.

- Assume that $B_{4R}(\mathbf{x}_0) \subset \Omega$ and $\omega \subset B_R(\mathbf{x}_0)$

For each real number $\Lambda > 0$ and each sequence $\{a_j\}_{j\geq 1} \in \ell^2$, we define

$$\mathbf{u}_\Lambda(\mathbf{x}) = \sum_{\lambda_j \leq \Lambda} a_j \mathbf{e}_j(\mathbf{x}), \quad \mathbf{x} \in \Omega,$$

and

$$\mathbf{v}_\Lambda(\mathbf{x}, s) = \sum_{\lambda_j \leq \Lambda} a_j e^{s\sqrt{\lambda_j}} d\mathbf{e}_j(\mathbf{x}), \quad (\mathbf{x}, s) \in \Omega \times (-1, 1),$$

where d denotes the curl operator.[5]
Because $\mathbf{v}_\Lambda(\cdot, 0) = d\mathbf{u}_\Lambda$ and $\mathrm{div}_{\mathbf{x}}\, \mathbf{u}_\Lambda = 0$, we have that

$$\Delta_{\mathbf{x}} \mathbf{u}_\Lambda(\mathbf{x}) = d^* \mathbf{v}_\Lambda(\mathbf{x}, 0), \quad \mathbf{x} \in \Omega,$$

where d^* is the adjoint of d.

Goal: to estimate the propagation of smallness for $\mathbf{u}_\Lambda$ on measurable sets with positive measure.

According to **Vessella's result**, it is sufficient to quantify the analytic estimates of higher-order derivatives of $\mathbf{u}_\Lambda$.

[5] In fact, d is the differential which maps 1-forms into 2-forms. When a vector field $\mathbf{w}$ is identified with a 1-form, then $d\mathbf{w}$ can be identified with a $\frac{1}{2}N(N-1)$-dimensional vector.

Notice that $\mathbf{v}_\Lambda(\cdot,\cdot)$ satisfies

$$-\partial^2_{ss}\mathbf{v}_\Lambda(\mathbf{x},s)-\Delta_{\mathbf{x}}\mathbf{v}_\Lambda(\mathbf{x},s)=0,\quad (\mathbf{x},s)\in\Omega\times(-1,1).$$

Since $d^*\mathbf{v}_\Lambda$ verifies

$$-\partial^2_{ss}d^*\mathbf{v}_\Lambda(\mathbf{x},s)-\Delta_{\mathbf{x}}d^*\mathbf{v}_\Lambda(\mathbf{x},s)=0,\quad (\mathbf{x},s)\in\Omega\times(-1,1),$$

the analyticity estimate for harmonic functions in $\mathbb{R}^{N+1}$ implies

$$\|\partial^\alpha_{\mathbf{x}}\partial^\beta_s d^*\mathbf{v}_\Lambda\|_{\mathbf{L}^\infty(B_{2R}(\mathbf{x}_0,0))}\leq C\frac{(|\alpha|+\beta)!}{(\rho R)^{|\alpha|+\beta}}\left(\int_{B_{4R}(\mathbf{x}_0,0)}|d^*\mathbf{v}_\Lambda(\mathbf{x},s)|^2d\mathbf{x}ds\right)^{1/2},\ \forall(\alpha,\beta)\in\mathbb{N}^{N+1}.$$

Taking $\beta=0$ in the previous estimate, we readily obtain

$$\|\partial^\alpha_{\mathbf{x}}d^*\mathbf{v}_\Lambda(\cdot,0)\|_{\mathbf{L}^\infty(B_{2R}(\mathbf{x}_0))}\leq C\frac{|\alpha|!}{(\rho R)^{|\alpha|}}\left(\int_{B_{4R}(\mathbf{x}_0,0)}|d^*\mathbf{v}_\Lambda(\mathbf{x},s)|^2d\mathbf{x}ds\right)^{1/2},\ \ \forall\alpha\in\mathbb{N}^N.$$

To bound the right-hand side, we set

$$\mathbf{w}_\Lambda(\mathbf{x},s) = \sum_{\lambda_j \le \Lambda} a_j e^{s\sqrt{\lambda_j}} \mathbf{e}_j(\mathbf{x}), \quad (\mathbf{x},s) \in \Omega \times (-1,1)$$

and then the following estimate holds

$$\|d^* \mathbf{v}_\Lambda\|^2_{\mathbf{L}^2(B_{4R}(\mathbf{x}_0,0))} \le C\|\mathbf{w}_\Lambda\|^2_{L^2((-1,1);\mathbf{H}^2(\Omega))} \le {}^6 C\|\mathbf{A}\mathbf{w}_\Lambda\|^2_{L^2((-1,1);\mathbf{H})} \le \ Ce^{C\sqrt{\Lambda}} \sum_{\lambda_j \le \Lambda} a_j^2,$$

for some $C > 0$.

Therefore, we have

$$\|\partial_x^\alpha d^* \mathbf{v}_\Lambda(\cdot,0)\|_{\mathbf{L}^\infty(B_{2R}(\mathbf{x}_0))} \le C\frac{|\alpha|!}{(\rho R)^{|\alpha|}} e^{C\sqrt{\Lambda}} (\sum_{\lambda_j \le \Lambda} a_j^2)^{1/2}, \ \ \forall \alpha \in \mathbb{N}^N,$$

where $C = C(N,\Omega)$.

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[6]It is well-known that there exists $C = C(N,\Omega) > 0$ such that

$$\frac{1}{C}\|\mathbf{y}\|_{\mathbf{H}^2(\Omega)} \le \|\mathbf{A}\mathbf{y}\|_{\mathbf{H}} \le C\|\mathbf{y}\|_{\mathbf{H}^2(\Omega)}, \quad \forall \mathbf{y} \in D(\mathbf{A}).$$

Since $\mathbf{u}_\Lambda$ solves the Poisson equation ($\Delta_{\mathbf{x}}\mathbf{u}_\Lambda = d^*\mathbf{v}_\Lambda(\cdot,0)$), we have that $\mathbf{u}_\Lambda$ is real-analytic whenever the exterior force $d^*\mathbf{v}_\Lambda(\cdot,0)$ is real-analytic. Moreover, the following estimate holds

$$\begin{aligned}
\|\partial_x^\alpha \mathbf{u}_\Lambda\|_{\mathbf{L}^\infty(B_R(\mathbf{x}_0))} &\leq \frac{|\alpha|!}{(R\tilde{\rho})^{|\alpha|+1}} \left(\|\mathbf{u}_\Lambda\|_{\mathbf{L}^2(B_{2R}(\mathbf{x}_0))} + Ce^{C\sqrt{\Lambda}} \left(\sum_{\lambda_j \leq \Lambda} a_j^2 \right)^{1/2} \right) \\
&\leq \frac{|\alpha|!}{(\rho R)^{|\alpha|}} e^{K\sqrt{\Lambda}} \left(\sum_{\lambda_j \leq \Lambda} a_j^2 \right)^{1/2}, \quad \forall \alpha \in \mathbb{N}^N,
\end{aligned}$$

where $\tilde{\rho}$, ρ and K are positive constants independent of Λ.

Using **Vessella's result**, applied to the real-analytic function $\mathbf{u}_\Lambda$, we obtain the estimate

$$\|\mathbf{u}_\Lambda\|_{\mathbf{L}^\infty(B_R(\mathbf{x}_0))} \le C \left(\int_\omega |\mathbf{u}_\Lambda(\mathbf{x})| \, d\mathbf{x} \right)^\theta \left(e^{K\sqrt{\Lambda}} \left(\sum_{\lambda_j \le \Lambda} a_j^2 \right)^{1/2} \right)^{1-\theta}$$

for some constants $C = C(N, R, \Omega, |\omega|) > 0$ and $\theta = \theta(N, R, \Omega, |\omega|) \in (0, 1)$.

Finally, the **spectral inequality obtained by Chaves-Silva and Lebeau** give us the desired observability inequality

$$\left(\sum_{\lambda_j \le \Lambda} a_j^2 \right)^{1/2} \le C e^{C\sqrt{\Lambda}} \int_\omega |\mathbf{u}_\Lambda(\mathbf{x})| \, d\mathbf{x}.$$

$\square$

Time optimal control problem for the Stokes system

For $r \in [1, \infty]$ and $M > 0$, consider the *set of admissible controls*

$$\mathcal{U}_{ad}^{M,r} = \{\mathbf{v} \in \mathbf{L}^{\infty}(\omega \times [0,\infty))\,;\, |\mathbf{v}(\mathbf{x},t)|_r \leq M \text{ a.e. in } \omega \times [0,\infty)\}^7$$

and define the *set of reachable states starting from* $\mathbf{u}_0$:

$$\mathcal{R}(\mathbf{u}_0, \mathcal{U}_{ad}^{M,r}) = \left\{\mathbf{u}(\cdot,\tau)\,;\, \tau > 0 \text{ and } \mathbf{u} \text{ solves the controlled Stokes system with } \mathbf{v} \in \mathcal{U}_{ad}^{M,r}\right\},$$

where the controlled Stokes system is:

$$\left|\begin{array}{lll} \mathbf{u}_t - \Delta \mathbf{u} + \nabla p = \mathbf{v}\chi_\omega & \text{in} & Q, \\ \operatorname{div} \mathbf{u} = 0 & \text{in} & Q, \\ \mathbf{u} = \mathbf{0} & \text{on} & \Sigma, \\ \mathbf{u}(\cdot,0) = \mathbf{u}_0 & \text{in} & \Omega. \end{array}\right.$$

Remark: $\mathbf{0} \in \mathcal{R}(\mathbf{u}_0, \mathcal{U}_{ad}^{M,r}) \quad \forall \mathbf{u}_0 \in \mathbf{H}$.

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[7] $|\cdot|_r : \mathbb{R}^N \to [0,\infty)$ is the r-euclidean norm in $\mathbb{R}^N$

Time optimal control problem

Time optimal control problem (TOCP):

given $\mathbf{u}_0 \in \mathbf{H}$ *and* $\mathbf{u}_f \in \mathcal{R}(\mathbf{u}_0, \mathcal{U}_{ad}^{M,r})$*, find* $\mathbf{v}_r^\star \in \mathcal{U}_{ad}^{M,r}$ *such that the corresponding solution* $\mathbf{u}^\star$ *of Stokes system satisfies*

$$\mathbf{u}^\star(\tau_r^\star(\mathbf{u}_0, \mathbf{u}_f)) = \mathbf{u}_f,$$

where

$$\tau_r^\star(\mathbf{u}_0, \mathbf{u}_f) = \min_{\mathbf{v} \in \mathcal{U}_{ad}^{M,r}} \{\tau\,;\, \mathbf{u}(\cdot, \tau) = \mathbf{u}_f\}\,.$$

- $\mathbf{v}_r^\star$ is the optimal control;
- $\tau_r^\star(\mathbf{u}_0, \mathbf{u}_f)$ is the optimal time.

Existence for time optimal control problem

Theorem

Let $M > 0$ and $r \in [1, \infty]$ be given. For every $\mathbf{u}_0 \in \mathbf{H}$ and any $\mathbf{u}_f \in \mathcal{R}(\mathbf{u}_0, \mathcal{U}_{ad}^{M,r})$, the time optimal problem has at least one solution. Moreover, any optimal control $\mathbf{v}_r^\star$ satisfies the bang-bang property: $|\mathbf{v}_r^\star(\mathbf{x}, t)|_r = M$ for a.e. $(\mathbf{x}, t) \in \omega \times [0, \tau_r^\star(\mathbf{u}_0, \mathbf{u}_f)]$.

Proof.

- $\exists (\tau_n, \mathbf{v}_n)_{n \geq 1}$ minimizing sequence such that:

  $$\tau_n \xrightarrow[n\to\infty]{} \tau_r^\star(\mathbf{u}_0, \mathbf{u}_f) \quad \text{and} \quad \mathbf{v}_n \to \mathbf{v}^\star \quad \text{weakly-}\star \ \text{ in } \ \mathbf{L}^\infty(\omega \times (0, \tau_r^\star(\mathbf{u}_0, \mathbf{u}_f)))$$

  with $(\mathbf{v}_n)_{n \geq 1} \subset \mathcal{U}_{ad}^{M,r}$ having the property that the associated solution $\mathbf{u}_n$ to Stokes system satisfies $\mathbf{u}_n(\cdot, \tau_n) = \mathbf{u}_f$ for all $n \geq 1$;
- $\mathbf{v}^\star$ is a solution of the time optimal problem;
- $\mathbf{v}^\star \in \mathcal{U}_{ad}^{M}$ is bang-bang (by contradiction);
  - Assume that there exist $\varepsilon > 0$ and a measurable set of positive measure $\gamma \subset \omega \times (0, \tau_r^\star(\mathbf{u}_0, \mathbf{u}_f))$ such that $|\mathbf{v}^*(\mathbf{x}, t)|_r < M - \varepsilon$ for $(\mathbf{x}, t) \in \gamma$.
  - Then, thanks to the $\mathbf{L}^\infty$-null controllability for Stokes system on measurable sets of positive measure, we can construct $\widehat{\mathbf{v}} \in \mathcal{U}_{ad}^{M,r}$ which is a control steering $\mathbf{u}_0$ to $\mathbf{u}_f$ at time $(\tau_r^\star(\mathbf{u}_0, \mathbf{u}_f) - \delta)$ for some $\delta > 0$.

$\square$

Uniqueness for time optimal control problem

Theorem

For every $\mathbf{u}_0 \in \mathbf{H}$ *and* $\mathbf{u}_f \in \mathcal{R}(\mathbf{u}_0, \mathcal{U}_{ad}^{M,r})$ *and for* $r \in (1, \infty)$, *the time optimal problem has a unique solution* $\mathbf{v}_r^\star$ *which is of bang-bang.*

Proof.

- assume that $\mathbf{v}$ and $\mathbf{h}$ are two time optimal controls in $\mathcal{U}_{ad}^{M,r}$;
- $\mathbf{w} = \frac{1}{2}(\mathbf{v} + \mathbf{h})$ is also an optimal control and satisfies the bang-bang property;
- use the fact that the norms $|\cdot|_r$ for $r \in (1, \infty)$ are *uniformly convex* in $\mathbb{R}^N$;

$\square$

Final comments

- $\mathbf{L}^{\infty}$- null controllability on measurable sets of positive measure;
- **Boundary spectral inequality** for Stokes?
- $r_1 \neq r_2$: are the optimal controls $\mathbf{v}^{\star}_{r_1}$ and $\mathbf{v}^{\star}_{r_2}$ related?
- $r = \infty$: uniqueness for the TOCP?

Thank you for your attention

and

Happy birthday Piermarco!!!