Observability and controllability properties for waves

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(\(M, g\)) be a smooth \(d\)-dimensional Riemannian manifold

\(\Delta_g\) Laplace-Beltrami operator on \(M\), associated with the metric \(g\)

\(\Omega\) open bounded connected subset of \(M\), with a smooth boundary if \(\partial\Omega \neq \emptyset\)

Consider the wave equation

\[
\partial_t^2 u - \Delta_g u = 0 \quad \text{in} \quad \mathbb{R} \times \Omega
\]

with Dirichlet or Neumann boundary conditions if \(\partial\Omega \neq \emptyset\).

Results hereafter are valid for more general time-independent wave operators

\[
\partial_t^2 - \sum_{i,j} a_{ij}(x) \partial_{x_i} \partial_{x_j} + \text{smooth lower-order terms}
\]
We have **observability on** \( Q \) **in time** \( T \) if there exists \( C > 0 \) such that

\[
C \| (u(0), \partial_t u(0)) \|_{H^0_0 \times L^2}^2 \leq \| \chi_Q \partial_t u \|_{L^2((0,T) \times \Omega)}^2 = \int_0^T \int_{\omega(t)} |\partial_t u(t,x)|^2 \, dx \, dt,
\]

\[
\Leftrightarrow C \| (u(0), \partial_t u(0)) \|_{L^2 \times H^{-1}}^2 \leq \| \chi_Q u \|_{L^2((0,T) \times \Omega)}^2 = \int_0^T \int_{\omega(t)} |u(t,x)|^2 \, dx \, dt
\]

(\text{Dirichlet case}) for any solution \( u \) of the wave equation.
Usual (static) Geometric Control Condition

Theorem (Bardos Lebeau Rauch, SICON 1992)

Take $\omega(t) \equiv \omega$ open. Under the GCC on $(\omega, T)$:

*Every geometric ray propagating in $\Omega$, and reflecting on $\partial \Omega$ according to the optics laws, meets $\omega$ within time $T$,*

and if moreover there is no ray having a contact of infinite order with $\partial \Omega$ whenever $\partial \Omega \neq \emptyset$, then we have observability on $\omega$ in time $T$. 
Now, $\omega(t)$ moves.

Le Rousseau Lebeau Terpolilli Trélat, APDE 2017

Take $Q$ open subset of $\mathbb{R} \times \overline{\Omega}$. Under the $t$-GCC on $(Q, T)$:

Every generalized bicharacteristic $s \mapsto (t(s), x(s), \tau(s), \xi(s))$ is such that there exists $s \in \mathbb{R}$ such that $t(s) \in (0, T)$ and $(t(s), x(s)) \in Q$, and if moreover there is no generalized bicharacteristic having a contact of infinite order with $(0, T) \times \partial \Omega$ whenever $\partial \Omega \neq \emptyset$, then we have observability on $Q$ in time $T$.

Remarks

- 1D case: Castro Münch Cindea, SICON 2014.
- Motivation of the study for the Total company.
Consequences:

1. Observability with few sensors

Take $Q$ open with Lipschitz boundary, with $(Q, T)$ satisfying $t$-GCC. Then every open subset $\nu$ of $[0, T] \times \overline{\Omega}$ (for the topology induced by $\mathbb{R} \times M$) containing $\partial (Q \cap ([0, T] \times \overline{\Omega})$ is such that $(\nu, T)$ satisfies $t$-GCC, and thus observability holds for $(\nu, T)$.

(not true for the usual GCC!)
Consequences:

2. Controllability

Under the previous assumptions, by duality, the wave equation with (time-dependent) internal control

\[ \partial_t^2 u - \triangle g u = \chi_Q f \]

is exactly controllable in \( H_0^1 \times L^2 \).
Time-varying control domains

Time-dependent Geometric Control Condition

Consequences:

3. Stabilization

There exist $\mu \geq 0$ and $\nu > 0$ such that any solution of

$$\partial^2_t u - \triangle g u + \chi_\omega(t) \partial_t u = 0$$

satisfies

$$\|u(t)\|_{H^1_0}^2 + \|\partial_t u(t)\|_{L^2}^2 \leq \mu \left( \|u(0)\|_{H^1_0}^2 + \|\partial_t u(0)\|_{L^2}^2 \right) e^{-\nu t}$$
We follow the lines of the classical proof by Bardos Lebeau Rauch:

1\textsuperscript{st} step: weak observability inequality

There exists $C > 0$ such that

$$C \| (u(0), \partial_t u(0)) \|_{H_0^1 \times L^2}^2 \leq \| \chi_Q \partial_t u \|_{L^2((0,T) \times \Omega)}^2 + \| (u(0), \partial_t u(0)) \|_{L^2 \times H^{-1}}^2$$

for any solution.

Proof by contradiction + propagation of singularities for defect measures. $t$-GCC is used to prove that the QL vanishes identically.
2\textsuperscript{nd} step: no invisible solution

Set of invisible solutions:

\[ N_T = \{ v \in H^1((0, T) \times \Omega) \mid v \text{ wave solution} \]  
with \( v(0) \in H^1_0, \partial_t v(0) \in L^2 \) and \( \chi_Q \partial_t v = 0 \}, \]

equipped with the norm \( \| v \|_{N_T}^2 = \| v(0) \|_{H^1_0}^2 + \| \partial_t v(0) \|_{L^2}^2 \).

\( N_T \) is closed.

We have \( N_T = \{ 0 \}. \)

→ The main simplification with respect to BLR 1992 is there.
Sketch of proof

1. Propagation of singularities + t-GCC
   \[ \Rightarrow \text{any invisible solution is smooth on } (0, T) \times \Omega \text{ up to the boundary} \]
   \[ \partial_t^2 - \triangle_g \text{ time-independent} \]
   \[ \Rightarrow N_T \text{ invariant under } \partial_t: \quad \nu \in N_T \Rightarrow \partial_t \nu \in N_T \]

2. weak observability
   \[ \Rightarrow C\|\nu\|_{N_T}^2 = C\|(v(0), \partial_t v(0))\|_{H_0^1 \times L^2} \leq \|(v(0), \partial_t v(0))\|_{L^2 \times H^{-1}} \quad \forall \nu \in N_T \]

Since \( H_0^1 \times L^2 \) is compactly embedded into \( L^2 \times H^{-1} \), this implies that the unit ball of \( N_T \) is compact and thus \( N_T \) is finite dimensional.

By contradiction: if \( N_T \neq \{0\} \), then \( \partial_t : N_T \rightarrow N_T \) has a (complex) eigenvalue \( \lambda \), with eigenfunction \( \nu \in N_T \setminus \{0\} : \nu(t, x) = e^{\lambda t} w(x) \Rightarrow (\lambda^2 - \triangle_g) w = 0 \).

Take \( t \) s.t. \( \omega(t) \neq \emptyset \). Since \( \chi_Q \partial_t \nu = 0 \) and thus \( \chi_Q \nu = 0 \), it follows that \( w = 0 \) on \( \omega(t) \).

By elliptic unique continuation: \( w = 0 \) on \( \Omega \), and hence \( \nu = 0 \). Contradiction.
In what follows:

\[ T_0(Q, \Omega) = \inf\{ T > 0 \mid (Q, T) \text{ satisfies } t\text{-GCC} \} \]

\[ \rightarrow \text{ minimal } t\text{-GCC time} \]
Time-varying control domains

Example: 1D

\[ M = \mathbb{R} \text{ (Euclidean), } \Omega = (0, 1) \]

\[ \omega(t) = (vt, vt + a) \text{ when } t \in (0, (1 - a)/v) \]

Then:

\[
T_0(v, a, \delta) = \begin{cases} 
  2(1 - a)/(1 + v) & \text{if } 0 \leq v < 1 \text{ and } \delta \geq 0, \\
  1 - a & \text{if } v = 1 \text{ and } \delta > 0, \\
  (1 - a)(3v + 1)/(v(1 + v)) & \text{if } v \geq 1 \text{ and } \delta = 0, \\
  (2(1 - a) + v\delta)(1 + v) & \text{if } v > 1 \text{ and } \delta > 0.
\end{cases}
\]
Example: moving domain on the sphere

\[ M = \Omega = S^2 \text{ (Euclidean)} \]

\[ a \in (0, 2\pi), \varepsilon \in (0, \pi/2), \nu > 0 \]

\[ \omega(t) = \{ (\theta, \varphi) \mid |\varphi| < \varepsilon, \nu t < \theta < \nu t + a \} \]

Then:

\[ T_0(\nu, a, \varepsilon) < +\infty \text{ except for a finite number of critical speeds } \nu > 0. \text{ Moreover:} \]

- \[ T_0(\nu, a, \varepsilon) \sim \frac{\pi - a}{\nu} \text{ as } \nu \to 0. \]
- If \( \nu > (2\pi - a + 2\varepsilon)/(2\varepsilon) \) then \( T_0(\nu, a, \varepsilon) < \infty. \)
  - If \( \nu \to +\infty \) then \( T_0(\nu, a, \varepsilon) \to \pi - 2\varepsilon. \)

Besides, if \( \nu \in \mathbb{Q} \), then there exist \( a_0 > 0 \) and \( \varepsilon_0 > 0 \) such that \( T_0(\nu, a, \varepsilon) = +\infty \) for every \( a \in (0, a_0) \) and every \( \varepsilon \in (0, \varepsilon_0). \)
Example: moving domain near the boundary of the disk

\[ M = \mathbb{R}^2 \text{ (Euclidean)}, \quad \Omega = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\} \]

\[ a \in (0, 2\pi), \quad \varepsilon \in (0, 1) \]

\[ \omega(t) = \{(r, \theta) \in [0, 1] \times \mathbb{R} \mid 1 - \varepsilon < r < 1, \ vt < \theta < vt + a\} \]
Time-varying control domains

Example: moving domain near the boundary of the disk

\[ T_0(v, a, \varepsilon) < +\infty, \text{ for every } v > (2\pi + 2\varepsilon - a)/(2\varepsilon) \]

\[ T_0(v, a, \varepsilon) \sim 2 - 2\varepsilon \text{ as } v \to +\infty. \]
Time-varying control domains

Example: moving domain near the boundary of the disk

If there exists \( n \in \mathbb{N} \setminus \{0, 1\} \) such that \( v \sin \frac{\pi}{n} \in \pi \mathbb{Q} \), then there exist \( a_0 \in (0, 2\pi) \) and \( \varepsilon_0 \in (0, 1) \) such that \( T_0(v, a, \varepsilon) = +\infty \) \( \forall a \in (0, a_0) \) \( \forall \varepsilon \in (0, \varepsilon_0) \).
Example: moving domain near the boundary of the disk

\[ \forall \nu \geq 1 \quad \forall a \in (0, 2\pi) \quad \exists \varepsilon_0 > 0 \text{ s.t. } T_0(\nu, a, \varepsilon) = +\infty \quad \forall \varepsilon \in (0, \varepsilon_0) \]

“secular effect”
Example: moving domain near the boundary of the disk

∀ v ≥ 0  ∀ a ∈ (0, π)  ∃ ε₀ ∈ (0, 1) s.t.  
T₀(v, a, ε) = +∞  ∀ ε ∈ (0, ε₀)
Even if \( a \approx 2\pi \), there exist \( v > 1 \) and \( \varepsilon > 0 \) small s.t. \( t \)-GCC fails, whereas GCC would be satisfied in the static case!
Other considerations

- Example of the square... (arithmetical considerations)

- Arbitrary domain: Do there exist $T > 0$ and an admissible $C^1$ path $t \mapsto x(t)$ in $\Omega$, with speed $\leq v$, such that $(Q, T)$ satisfies $t$-GCC?

- Observation domain or control domain on the boundary: similar results

J. Le Rousseau, G. Lebeau, P. Terpolilli, E. Trélat,

*Geometric control condition for the wave equation with a time-dependent domain*,
Turn back to static control or observation domains.

**Geometric quantity**

\[ g_2(\omega) = \lim_{T \to +\infty} \inf_{\gamma} \frac{1}{T} \int_0^T \chi_\omega(\gamma(t)) \, dt, \]

where \( \gamma \) runs over all rays.

**Spectral quantity**

\[ g_1(\omega) = \inf_{\phi \in \mathcal{E}} \int_\omega \phi^2 \, dx, \]

where \( \mathcal{E} \) = set of normalized Laplacian eigenfunctions.
Generalized GCC

For every $\omega$ measurable, we have

$$g^T_2 (\hat{\omega}) > 0 \implies C_T (\bar{\omega}) \geq C_T (\omega) \geq C_T (\hat{\omega}) > 0 \quad \text{(usual GCC)}$$
$$C_T (\bar{\omega}) > 0 \implies g^T_2 (\bar{\omega}) > 0$$

In particular, if $\omega$ has no grazing ray then $g^T_2 (\hat{\omega}) = g^T_2 (\bar{\omega}) = g^T_2 (\omega)$, and then we have the equivalence

$$g^T_2 (\omega) > 0 \iff C_T (\omega) > 0$$

Time asymptotic observability constant

Given any $\omega$ measurable with no grazing ray, we have

$$\lim_{{T \to +\infty}} \frac{C_T (\omega)}{T} = \frac{1}{2} \min (g_1 (\omega), g_2 (\bar{\omega}))$$

(compare with Lebeau's result for damped wave equations)
Main new idea here:

- Define $C_T^N(\omega)$, “high-frequency observability constant”

- Define $\alpha_T(\omega) = \lim_{N \to +\infty} \frac{1}{T} C_T^N(\omega) = \sup_N \frac{1}{T} C_T^N(\omega)$

- Prove that $$\lim_{T \to +\infty} \frac{C_T(\omega)}{T} = \min \left( \frac{1}{2} g_1(\omega), \limsup_{T \to +\infty} \alpha_T(\omega) \right)$$

- Prove that $C_T(\omega) > 0 \iff \alpha_T(\omega) > 0 \ \forall \omega$ measurable (elementary proof by considering invisible solutions).

E. Humbert, Y. Privat, E. Trélat,
*Observability properties of the homogeneous wave equation on a closed manifold*, preprint (2016).

Still ongoing works using this “quantum observability constant” $\alpha_T(\omega)$. 
Characterization of Zoll manifolds

Known results:

- $X$ Hamiltonian vector field (on $S^*M$) of $\sqrt{\Delta}$
- $S = \frac{1}{i} L_X$
- $\Sigma =$ closure of $\{\lambda - \mu \mid \lambda, \mu \in \text{Spec}\sqrt{\Delta}\}$

Helton, Guillemin, 1977

- $\text{Spec}(S) \subset \Sigma$.
- If there exists a nonperiodic geodesic, then $\text{Spec}(S) = \mathbb{R}$, and thus $\Sigma = \mathbb{R}$.
- $M$ is Zoll if and only if $\Sigma \neq \mathbb{R}$.
  In this case, we have $\Sigma = \frac{2\pi}{T} \mathbb{Z}$, where $T$ is the smallest common period.
Characterization of Zoll manifolds

New results:

Geometric quantities

\[ g_2(\omega) = \liminf_{T \to +\infty} \inf_{\gamma} \frac{1}{T} \int_0^T \chi_{\omega}(\gamma(t)) \, dt, \quad g'_2(\omega) = \inf_{\gamma} \liminf_{T \to +\infty} \frac{1}{T} \int_0^T \chi_{\omega}(\gamma(t)) \, dt \]

where \( \gamma \) runs over all rays. Note that \( g_2 \leq g'_2 \).

Spectral quantities

\[ g_1(\omega) = \inf_{\phi \in \mathcal{E}} \int_{\omega} \phi^2 \, dx, \quad g'_1(\omega) = \inf_{\mu \in QL} \mu(\omega) \]

where \( \mathcal{E} = \) set of normalized Laplacian eigenfunctions, and \( QL = \) quantum limits. Note that \( g_1(\omega) \leq g'_1(\omega) \) for any closed \( \omega \).

For \( \gamma \) \( T \)-periodic ray on \( M \): Dirac measure \( \delta_{\gamma}(f) = \frac{1}{T} \int_0^T f(\gamma(t)) \, dt \quad \forall f \in C^0(M) \).
Characterization of Zoll manifolds

New results:

Humbert Privat Trélat, ongoing

- $M$ Zoll and $\delta_\gamma \in QL$ $\forall \gamma$ periodic geod. $\iff g_1(\omega) \leq g_2(\omega)$ $\forall \omega$ closed

- $M$ Zoll $\iff g_2(\omega) = g'_2(\omega)$ $\forall \omega$ closed $\rightarrow$ “Zoll defect” $Z(\omega) = g'_2(\omega) - g_2(\omega)$

- $M$ Zoll, two-point homogeneous and $\delta_\gamma \in QL$ $\forall \gamma$ periodic geod. $\Rightarrow QL = I$.

- Spectral gap $\Rightarrow M$ Zoll, $\delta_\gamma \in QL$ $\forall \gamma$ periodic geod. $\Rightarrow QL = I$ (cf also Macia)

- Uniformly locally finite spectrum (i.e., $\exists \ell > 0$ and $m \in \mathbb{N}^*$ s.t. the intersection of the spectrum with any interval of length $\ell$ has at most $m$ distinct elements) $\Rightarrow M$ Zoll, and $\forall \gamma$ geodesic $\exists \mu \in QL$ s.t. $\mu(\gamma(\cdot)) > 0$.

“Zollditch” conjecture: $QL = I \Rightarrow M$ Zoll
Sufficient condition for Schrödinger internal observability

New results:

\[ g'_2(\omega) > 0 \implies C^{\text{Schrod}}_T(\omega) > 0 \]

- More general than the usual GCC condition \( g_2(\omega) > 0 \)
- Not a necessary condition: \( M = T^2 \)
- Actually, \( C^{\text{Schrod}}_T(\omega) > 0 \iff \alpha^{T,\text{Schrod}}(\omega) > 0 \), and \( \alpha^{T,\text{Schrod}}(\omega) \geq g'_2(\omega) \geq g_2(\omega) \)

As a corollary:

If \( M \) is not Zoll, then \( \forall T > 0 \exists \omega \text{ open s.t. } C^\text{wave}_T(\omega) = 0 \) and \( C^{\text{Schrod}}_T(\omega) > 0 \)

E. Humbert, Y. Privat, E. Trélat, ongoing.
Time-varying control domains

E. Trélat

Observability and controllability properties for waves
Microlocal interpretation

\[ g_2^T(a) = \inf_{z \in S^* M} \frac{1}{T} \int_0^T a \circ \varphi_t(z) \, dt = \inf_{z \in S^* M} \bar{a}_T(z) \quad \forall a \in L^\infty(S^* M, \mu_L) \]

\[ g_2(a) = \lim_{T \to +\infty} \inf_{z \in S^* M} \frac{1}{T} \int_0^T a \circ \varphi_t(z) \, dt = \lim_{T \to +\infty} \inf_{z \in S^* M} \bar{a}_T(z) \]

\[ g'_2(a) = \inf_{z \in S^* M} \lim_{T \to +\infty} \frac{1}{T} \int_0^T a \circ \varphi_t(z) \, dt = \inf_{z \in S^* M} \lim_{T \to +\infty} \bar{a}_T(z) \]
Microlocal interpretation

\[ a_t = a \circ \varphi_t = \sigma_P(A_t) \quad \text{with} \quad A_t = e^{-it\sqrt{\Delta}} Ae^{it\sqrt{\Delta}}, \quad A = \text{Op}(a) \in \Psi^0(M) \]

\[ \bar{a}_T = \sigma_P(\bar{A}_T) \quad \text{with} \quad \bar{A}_T = \frac{1}{T} \int_0^T A_t \, dt = \frac{1}{T} \int_0^T e^{-it\sqrt{\Delta}} Ae^{it\sqrt{\Delta}} \, dt. \]

Set \( f_T(t) = ie^{iTt/2}\text{sinc}(Tt/2) \), we have \( \hat{f}_T(t) = \frac{1}{T} \chi_{[0,T]}(t) \). Then

\[ g_2^T(a) = \inf_{z \in S^*M} \frac{1}{T} \int_0^T a \circ e^{tX}(z) \, dt = \inf_{z \in S^*M} \int_R \hat{f}_T(t)e^{itS} a \, dt (z) = \inf f_T(S)a \]

\[ g_2'(a) = \inf_{S^*M} \liminf_{T \to +\infty} f_T(S)a = \inf Q_0 a \quad (Q_0 = \text{eigenprojection onto} \ \ker S) \]

\[ C_T(a) = \inf_{\|y\|=1} \langle \bar{A}_T(a)y, y \rangle = \inf_{\|y\|=1} \langle A_f y, y \rangle \quad \text{(half-waves)} \]