Observability and controllability properties for waves

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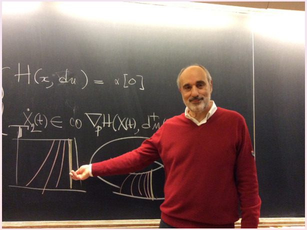
New Trends in Control Theory and PDEs, July 2017
Piermarco Cannarsa's Birthday



Setting

- (M, g) be a smooth d-dimensional Riemannian manifold
- $\triangle_g$ Laplace-Beltrami operator on M, associated with the metric g
- Ω open bounded connected subset of M, with a smooth boundary if $\partial\Omega \neq \emptyset$

Consider the wave equation

$$\boxed{\partial_t^2 u - \triangle_g u = 0 \qquad \text{in} \quad \mathbb{R} \times \Omega}$$

with Dirichlet or Neumann boundary conditions if $\partial\Omega \neq \emptyset$.

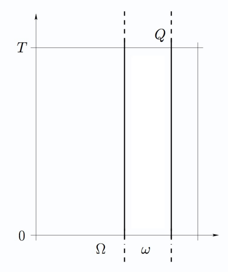
Results hereafter are valid for more general time-independent wave operators

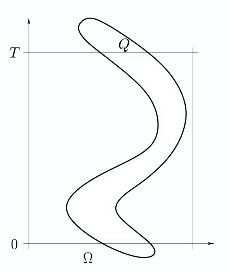
$$\partial_t^2 - \sum_{i,j} a_{ij}(x)\partial_{x_i}\partial_{x_j} + \text{smooth lower-order terms}$$

Observability with a time-varying domain

- Q open subset of $\mathbb{R} \times \overline{\Omega}$
- $\omega(t) = \{x \in \Omega \mid (t,x) \in Q\}$, i.e.,

$$Q = \{(t,x) \in \mathbb{R} \times \Omega \mid t \in \mathbb{R},\ x \in \omega(t)\}$$





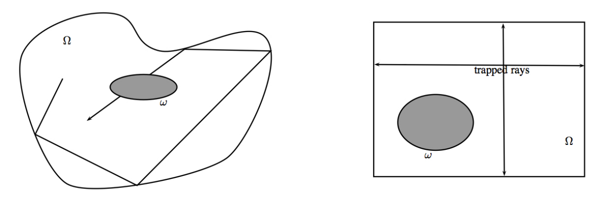
We have observability on Q in time T if there exists $C > 0$ such that

$$C\|(u(0),\partial_t u(0))\|^2_{H^1_0\times L^2} \leqslant \|\chi_Q \partial_t u\|^2_{L^2((0,T)\times\Omega)} = \int_0^T \int_{\omega(t)} |\partial_t u(t,x)|^2 \, dx_g \, dt,$$

$$\Leftrightarrow C\|(u(0),\partial_t u(0))\|^2_{L^2\times H^{-1}} \leqslant \|\chi_Q u\|_{L^2((0,T)\times\Omega)}\|^2 = \int_0^T \int_{\omega(t)} |u(t,x)|^2 \, dx_g \, dt$$

(Dirichlet case) for any solution u of the wave equation.

Usual (static) Geometric Control Condition



Theorem (Bardos Lebeau Rauch, SICON 1992)

Take $\omega(t) \equiv \omega$ open. Under the GCC on (ω, T):

> *Every geometric ray propagating in Ω, and reflecting on $\partial\Omega$ according to the optics laws, meets ω within time T,*

and if moreover there is no ray having a contact of infinite order with $\partial\Omega$ whenever $\partial\Omega \neq \emptyset$, then we have observability on ω in time T.

Time-dependent Geometric Control Condition

Now, $\omega(t)$ moves.

Le Rousseau Lebeau Terpolilli Trélat, APDE 2017

Take Q open subset of $\mathbb{R} \times \overline{\Omega}$. Under the t-GCC on (Q, T):

> *Every generalized bicharacteristic $s \mapsto (t(s), x(s), \tau(s), \xi(s))$ is such that there exists $s \in \mathbb{R}$ such that $t(s) \in (0, T)$ and $(t(s), x(s)) \in Q$,*

and if moreover there is no generalized bicharacteristic having a contact of infinite order with $(0, T) \times \partial\Omega$ whenever $\partial\Omega \neq \emptyset$, then we have observability on Q in time T.

Remarks

- 1D case: Castro Münch Cindea, SICON 2014.
- Motivation of the study for the Total company.

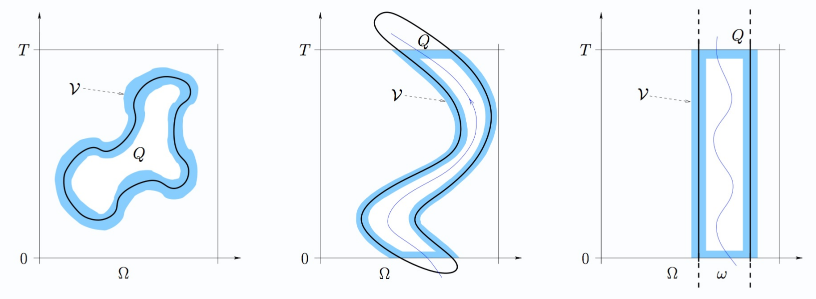
Time-dependent Geometric Control Condition

Consequences:

1. Observability with few sensors

Take Q open with Lipschitz boundary, with (Q, T) satisfying t-GCC.

Then every open subset $\mathcal{V}$ of $[0, T] \times \overline{\Omega}$ (for the topology induced by $\mathbb{R} \times M$) containing $\partial\left(Q \cap ([0, T] \times \overline{\Omega})\right)$ is such that $(\mathcal{V}, T)$ satisfies t-GCC, and thus observability holds for $(\mathcal{V}, T)$.



(a) (b) (c)

(not true for the usual GCC!)

Time-dependent Geometric Control Condition

Consequences:

2. Controllability

Under the previous assumptions, by duality, the wave equation with (time-dependent) internal control

$$\boxed{\partial_t^2 u - \triangle_g u = \chi_Q f}$$

is exactly controllable in $H_0^1 \times L^2$.

Time-dependent Geometric Control Condition

Consequences:

3. Stabilization

There exist $\mu \geqslant 0$ and $\nu > 0$ such that any solution of

$$\boxed{\partial_t^2 u - \triangle_g u + \chi_{\omega(t)} \partial_t u = 0}$$

satisfies

$$\|u(t)\|_{H_0^1}^2 + \|\partial_t u(t)\|_{L^2}^2 \leqslant \mu \left(\|u(0)\|_{H_0^1}^2 + \|\partial_t u(0)\|_{L^2}^2 \right) e^{-\nu t}$$

Sketch of proof

We follow the lines of the classical proof by Bardos Lebeau Rauch:

1st step: weak observability inequality

There exists $C > 0$ such that

$$C\|(u(0), \partial_t u(0))\|^2_{H^1_0 \times L^2} \leqslant \|\chi_Q \partial_t u\|^2_{L^2((0,T)\times\Omega)} + \|(u(0), \partial_t u(0))\|^2_{L^2 \times H^{-1}}$$

for any solution.

Proof by contradiction + propagation of singularities for defect measures.

t-GCC is used to prove that the QL vanishes identically.

Sketch of proof

2^{nd} step: no invisible solution

Set of *invisible solutions*:

$$N_T = \{ v \in H^1((0,T)\times\Omega) \ \mid \ v \text{ wave solution} \\ \text{with } v(0)\in H^1_0,\ \partial_t v(0) \in L^2 \ \text{ and } \chi_Q \partial_t v = 0 \},$$

equiped with the norm $\|v\|_{N_T}^2 = \|v(0)\|_{H^1_0}^2 + \|\partial_t v(0)\|_{L^2}^2$.

N_T is closed.

We have $N_T = \{0\}$.

$\rightarrow$ The main simplification with respect to BLR 1992 is there.

Sketch of proof

1.
   - Propagation of singularities + t-GCC
     $\Rightarrow$ any invisible solution is smooth on $(0,T)\times\Omega$ up to the boundary
   - $\partial_t^2 - \triangle_g$ time-independent

   $\Rightarrow$ N_T invariant under ∂_t: $\quad v\in N_T \Rightarrow \partial_t v\in N_T$

2. weak observability

   $$\Rightarrow\ C\|v\|_{N_T}^2 = C\|(v(0),\partial_t v(0))\|_{H_0^1\times L^2} \leqslant \|(v(0),\partial_t v(0))\|_{L^2\times H^{-1}} \quad \forall v\in N_T$$

   Since $H_0^1\times L^2$ is compactly embedded into $L^2\times H^{-1}$, this implies that the unit ball of N_T is compact and thus N_T is finite dimensional.

By contradiction: if $N_T\neq\{0\}$, then $\partial_t : N_T\to N_T$ has a (complex) eigenvalue λ, with eigenfunction $v\in N_T\setminus\{0\}$: $v(t,x)=e^{\lambda t}w(x)\Rightarrow(\lambda^2-\triangle_g)w=0$.

Take t s.t. $\omega(t)\neq\emptyset$. Since $\chi_Q\partial_t v=0$ and thus $\chi_Q v=0$, it follows that $w=0$ on $\omega(t)$.
By elliptic unique continuation: $w=0$ on Ω, and hence $v=0$. Contradiction.

Examples

In what follows:

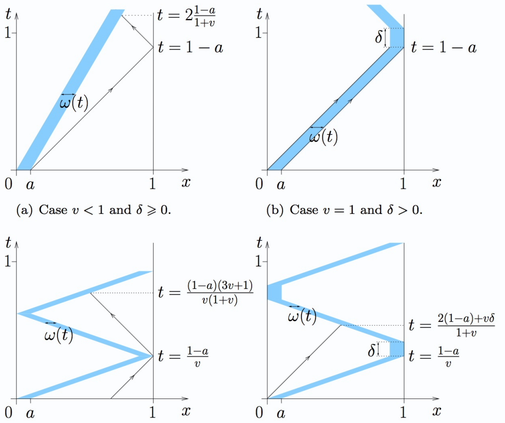
$$T_0(Q,\Omega) = \inf\{T > 0 \mid (Q,T) \text{ satisfies } t\text{-GCC}\}$$

$\rightarrow$ minimal t-GCC time

Example: 1D

$M = \mathbb{R}$ (Euclidean), $\Omega = (0,1)$

$\omega(t) = (vt, vt + a)$ when $t \in (0, (1-a)/v)$



(a) Case $v < 1$ and $\delta \geqslant 0$. (b) Case $v = 1$ and $\delta > 0$.

(c) Case $v \geqslant 1$ and $\delta = 0$. (d) Case $v > 1$ and $\delta > 0$.

Then:

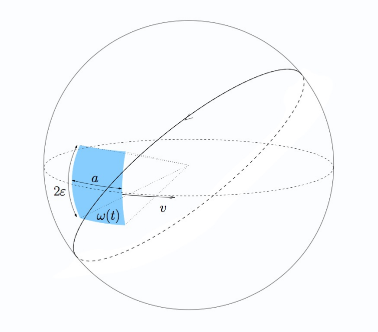
$$T_0(v, a, \delta) = \begin{cases} 2(1-a)/(1+v) & \text{if } 0 \leqslant v < 1 \text{ and } \delta \geqslant 0, \\ 1 - a & \text{if } v = 1 \text{ and } \delta > 0, \\ (1-a)(3v+1)/(v(1+v)) & \text{if } v \geqslant 1 \text{ and } \delta = 0, \\ \big(2(1-a) + v\delta\big)(1+v) & \text{if } v > 1 \text{ and } \delta > 0. \end{cases}$$

Example: moving domain on the sphere

$M = \Omega = S^2$ (Euclidean)

$a \in (0, 2\pi)$, $\varepsilon \in (0, \pi/2)$, $v > 0$

$$\omega(t) = \{(\theta, \varphi) \mid |\varphi| < \varepsilon,\ vt < \theta < vt + a\}$$



Then:

$T_0(v, a, \varepsilon) < +\infty$ except for a finite number of critical speeds $v > 0$. Moreover:

- $T_0(v, a, \varepsilon) \sim \frac{\pi - a}{v}$ as $v \to 0$.
- If $v > (2\pi - a + 2\varepsilon)/(2\varepsilon)$ then $T_0(v, a, \varepsilon) < \infty$.
  If $v \to +\infty$ then $T_0(v, a, \varepsilon) \to \pi - 2\varepsilon$.

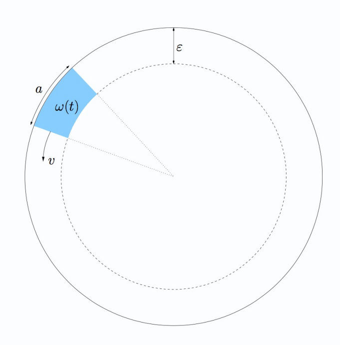
Besides, if $v \in \mathbb{Q}$, then there exist $a_0 > 0$ and $\varepsilon_0 > 0$ such that $T_0(v, a, \varepsilon) = +\infty$ for every $a \in (0, a_0)$ and every $\varepsilon \in (0, \varepsilon_0)$.

Example: moving domain near the boundary of the disk

$M = \mathbb{R}^2$ (Euclidean), $\Omega = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$

$a \in (0, 2\pi)$, $\varepsilon \in (0,1)$

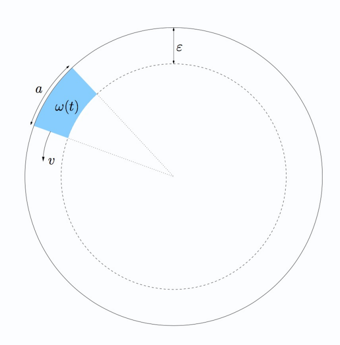
$\omega(t) = \{(r,\theta) \in [0,1] \times \mathbb{R} \mid 1 - \varepsilon < r < 1,\ vt < \theta < vt + a\}$



Example: moving domain near the boundary of the disk

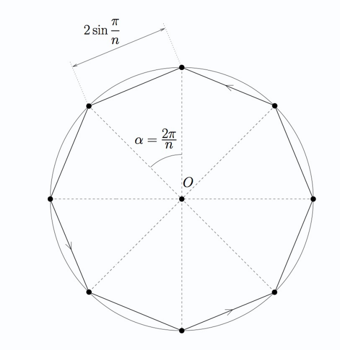
$T_0(v, a, \varepsilon) < +\infty$, for every $v > (2\pi + 2\varepsilon - a)/(2\varepsilon)$

$T_0(v, a, \varepsilon) \sim 2 - 2\varepsilon$ as $v \to +\infty$.



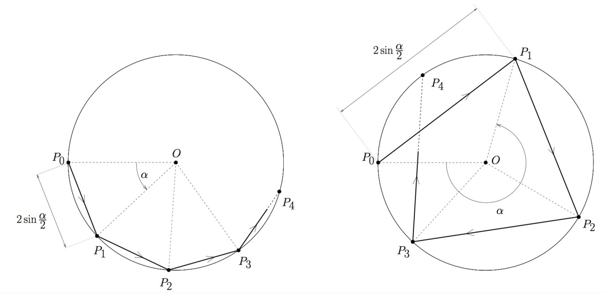
Example: moving domain near the boundary of the disk

If there exists $n \in \mathbb{N} \setminus \{0,1\}$ such that $v \sin \frac{\pi}{n} \in \pi\mathbb{Q}$,
then there exist $a_0 \in (0, 2\pi)$ and $\varepsilon_0 \in (0,1)$ such that
$T_0(v, a, \varepsilon) = +\infty \quad \forall a \in (0, a_0) \quad \forall \varepsilon \in (0, \varepsilon_0)$.



Example: moving domain near the boundary of the disk

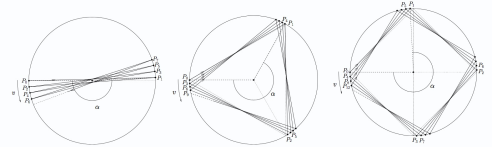
$\forall v \geqslant 1 \quad \forall a \in (0, 2\pi) \quad \exists \varepsilon_0 > 0$ s.t. $T_0(v, a, \varepsilon) = +\infty \quad \forall \varepsilon \in (0, \varepsilon_0)$



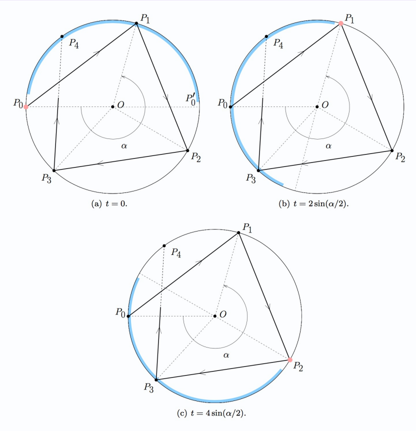
"secular effect"

Example: moving domain near the boundary of the disk

$\forall v \geqslant 0 \quad \forall a \in (0,\pi) \quad \exists \varepsilon_0 \in (0,1)$ s.t.
$T_0(v,a,\varepsilon) = +\infty \quad \forall \varepsilon \in (0,\varepsilon_0)$



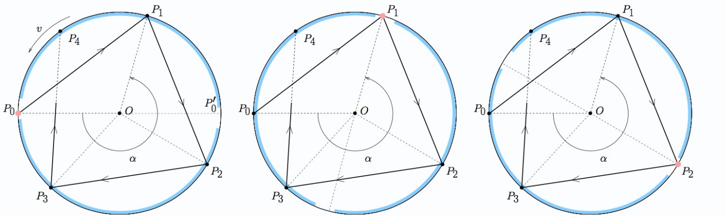
(a) $n=2$, $\pi < \alpha < \pi+\delta$. (b) $n=3$, $4\pi/3 < \alpha < 4\pi/3+\delta$. (c) $n=4$, $3\pi/2 < \alpha < 3\pi/2+\delta$.



(a) $t=0$. (b) $t = 2\sin(\alpha/2)$. (c) $t = 4\sin(\alpha/2)$.

FSMP Fondation Sciences Mathématiques de Paris

Example: moving domain near the boundary of the disk



(a) $t = 0$. (b) $t = 2\sin(\alpha/2)$. (c) $t = 4\sin(\alpha/2)$.

Even if $a \simeq 2\pi$, there exist $v > 1$ and $\varepsilon > 0$ small s.t. t-GCC fails, whereas GCC would be satisfied in the static case!

Other considerations

- Example of the square... (arithmetic considerations)
- Arbitrary domain: Do there exist $T > 0$ and an admissible C^1 path $t \mapsto x(t)$ in Ω, with speed $\leqslant v$, such that (Q, T) satisfies t-GCC?
- observation domain or control domain on the boundary: similar results

J. Le Rousseau, G. Lebeau, P. Terpolilli, E. Trélat,
Geometric control condition for the wave equation with a time-dependent domain,
Anal. PDE (2017).

Turn back to static control or observation domains.

Geometric quantity

$$g_2(\omega) = \liminf_{T\to+\infty} \underbrace{\inf_{\gamma} \frac{1}{T}\int_0^T \chi_\omega(\gamma(t))\,dt}_{g_2^T(\omega)},$$

where γ runs over all rays.

Spectral quantity

$$g_1(\omega) = \inf_{\phi\in\mathcal{E}} \int_\omega \phi^2\,dx,$$

where $\mathcal{E}$ = set of normalized Laplacian eigenfunctions.

Generalized GCC

For every ω measurable, we have

$$g_2^T(\mathring{\omega}) > 0 \;\Rightarrow\; C_T(\bar\omega) \geqslant C_T(\omega) \geqslant C_T(\mathring{\omega}) > 0 \qquad \text{(usual GCC)}$$
$$C_T(\bar\omega) > 0 \;\Rightarrow\; g_2^T(\bar\omega) > 0$$

In particular, if ω has no grazing ray then $g_2^T(\mathring{\omega}) = g_2^T(\bar\omega) = g_2^T(\omega)$, and then we have the equivalence

$$g_2^T(\omega) > 0 \Leftrightarrow C_T(\omega) > 0$$

Time asymptotic observability constant

Given any ω measurable with no grazing ray, we have

$$\lim_{T\to+\infty} \frac{C_T(\omega)}{T} = \frac{1}{2}\min\left(g_1(\omega), g_2(\overline{\omega})\right)$$

(compare with Lebeau's result for damped wave equations)

Main new idea here:

- Define $C_T^{>N}(\omega)$, “high-frequency observability constant”
- Define $\alpha^T(\omega) = \lim_{N\to+\infty} \frac{1}{T} C_T^{>N}(\omega) = \sup_N \frac{1}{T} C_T^{>N}(\omega)$
- Prove that $\lim_{T\to+\infty} \frac{C_T(\omega)}{T} = \min\left(\frac{1}{2} g_1(\omega), \limsup_{T\to+\infty} \alpha^T(\omega) \right)$
- Prove that $C_T(\omega) > 0 \Leftrightarrow \alpha^T(\omega) > 0 \quad \forall \omega$ measurable (elementary proof by considering invisible solutions).

E. Humbert, Y. Privat, E. Trélat,
Observability properties of the homogeneous wave equation on a closed manifold, preprint (2016).

Still ongoing works using this “quantum observability constant” $\alpha^T(\omega)$.

Characterization of Zoll manifolds

Known results:

- X Hamiltonian vector field (on S^*M) of $\sqrt{\triangle}$
- $S = \frac{1}{i} L_X$
- $\Sigma =$ closure of $\{\lambda - \mu \mid \lambda, \mu \in \mathrm{Spec}\sqrt{\triangle}\}$

Helton, Guillemin, 1977

- $\mathrm{Spec}(S) \subset \Sigma$.
- If there exists a nonperiodic geodesic, then $\mathrm{Spec}(S) = \mathbb{R}$, and thus $\Sigma = \mathbb{R}$.
- M is Zoll if and only if $\Sigma \neq \mathbb{R}$.
  In this case, we have $\Sigma = \frac{2\pi}{T}\mathbb{Z}$, where T is the smallest common period.

Characterization of Zoll manifolds

New results:

Geometric quantities

$$g_2(\omega) = \liminf_{T\to+\infty} \underbrace{\inf_{\gamma} \frac{1}{T}\int_0^T \chi_\omega(\gamma(t))\,dt}_{g_2^T(\omega)}, \qquad g_2'(\omega) = \inf_{\gamma} \liminf_{T\to+\infty} \frac{1}{T}\int_0^T \chi_\omega(\gamma(t))\,dt$$

where γ runs over all rays. Note that $g_2 \leqslant g_2'$.

Spectral quantities

$$g_1(\omega) = \inf_{\phi\in\mathcal{E}} \int_\omega \phi^2\,dx, \qquad g_1'(\omega) = \inf_{\mu\in QL} \mu(\omega)$$

where $\mathcal{E}$ = set of normalized Laplacian eigenfunctions, and QL = quantum limits. Note that $g_1(\omega) \leqslant g_1'(\omega)$ for any closed ω.

For γ T-periodic ray on M: Dirac measure $\delta_\gamma(f) = \frac{1}{T}\int_0^T f(\gamma(t))\,dt \quad \forall f \in C^0(M)$.

Characterization of Zoll manifolds

New results:

Humbert Privat Trélat, ongoing

- M Zoll and $\delta_\gamma \in QL \quad \forall\gamma$ periodic geod. $\Leftrightarrow g_1(\omega) \leqslant g_2(\omega) \quad \forall\omega$ closed
- M Zoll $\Leftrightarrow g_2(\omega) = g_2'(\omega) \quad \forall\omega$ closed $\qquad \rightarrow$ "Zoll defect" $Z(\omega) = g_2'(\omega) - g_2(\omega)$
- M Zoll, two-point homogeneous and $\delta_\gamma \in QL \quad \forall\gamma$ periodic geod. $\Rightarrow QL = \mathcal{I}$.
- Spectral gap $\Rightarrow M$ Zoll, $\quad \delta_\gamma \in QL \quad \forall\gamma$ periodic geod. (cf also Macia)
  $(g_1(\omega) \leqslant)\; g_1'(\omega) \leqslant g_2(\bar\omega) \quad \forall\omega$ measurable
- *Uniformly locally finite* spectrum (i.e., $\exists\ell > 0$ and $m \in \mathbb{N}^*$ s.t. the intersection of the spectrum with any interval of length ℓ has at most m *distinct* elements)
  $\Rightarrow M$ Zoll, and $\forall\gamma$ geodesic $\quad \exists\mu \in QL$ s.t. $\mu(\gamma(\cdot)) > 0$.

"Zollditch" conjecture: $QL = \mathcal{I} \Rightarrow M$ Zoll

Sufficient condition for Schrödinger internal observability

<u>New results</u>:

$$g_2'(\omega) > 0 \Rightarrow C_T^{\mathrm{Schrod}}(\omega) > 0$$

- More general than the usual GCC condition $g_2(\omega) > 0$
- Not a necessary condition: $M = \mathbb{T}^2$
- actually, $C_T^{\mathrm{Schrod}}(\omega) > 0 \Leftrightarrow \alpha^{T,\mathrm{Schrod}}(\omega) > 0$, and $\alpha^{T,\mathrm{Schrod}}(\omega) \geqslant g_2'(\omega) \geqslant g_2(\omega)$

As a corollary:

If M is not Zoll, then $\forall T > 0$ $\exists \omega$ open s.t. $C_T^{\mathrm{wave}}(\omega) = 0$ and $C_T^{\mathrm{Schrod}}(\omega) > 0$

E. Humbert, Y. Privat, E. Trélat, *ongoing*.

Microlocal interpretation

$$g_2^T(a) = \inf_{z\in S^*M} \frac{1}{T}\int_0^T a\circ\varphi_t(z)\,dt = \inf_{z\in S^*M} \bar{a}_T(z) \qquad \forall a\in L^\infty(S^*M,\mu_L)$$

$$g_2(a) = \liminf_{T\to+\infty} \underbrace{\inf_{z\in S^*M} \frac{1}{T}\int_0^T a\circ\varphi_t(z)\,dt}_{g_2^T(a)} = \liminf_{T\to+\infty} \inf_{z\in S^*M} \bar{a}_T(z)$$

$$g_2'(a) = \inf_{z\in S^*M} \liminf_{T\to+\infty} \frac{1}{T}\int_0^T a\circ\varphi_t(z)\,dt = \inf_{z\in S^*M} \liminf_{T\to+\infty} \bar{a}_T(z)$$

Microlocal interpretation

$$a_t = a \circ \varphi_t = \sigma_P(A_t) \qquad \text{with} \qquad A_t = e^{-it\sqrt{\triangle}} A e^{it\sqrt{\triangle}}, \quad A = \mathrm{Op}(a) \in \Psi^0(M)$$

$\bar{a}_T = \sigma_P(\bar{A}_T)$ with $\bar{A}_T = \frac{1}{T}\int_0^T A_t\, dt = \frac{1}{T}\int_0^T e^{-it\sqrt{\triangle}} A e^{it\sqrt{\triangle}}\, dt.$

Set $f_T(t) = ie^{iTt/2}\mathrm{sinc}(Tt/2)$, we have $\hat{f}_T(t) = \frac{1}{T}\chi_{[0,T]}(t)$. Then

- $g_2^T(a) = \inf\limits_{z \in S^*M} \dfrac{1}{T}\displaystyle\int_0^T a \circ e^{tX}(z)\, dt = \inf\limits_{z \in S^*M} \int_{\mathbb{R}} \hat{f}_T(t) e^{itS} a\, dt\, (z) = \inf f_T(S) a$
- $g_2'(a) = \inf\limits_{S^*M} \liminf\limits_{T \to +\infty} f_T(S) a = \inf Q_0 a \quad (Q_0 = \text{eigenprojection onto } \ker S)$
- $C_T(a) = \inf\limits_{\|y\|=1} \langle \bar{A}_T(a) y, y \rangle = \inf\limits_{\|y\|=1} \langle A_f y, y \rangle \qquad \text{(half-waves)}$