Carleman estimates for viscoelasticity equations and applications

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New Trends in Control Theory and PDEs
on the occasion of the 60th birthday of Professor Piermarco Cannarsa

INdAM - Rome 7 July 2017
First meeting (1990?) at Vorau (Steiermark)
First meeting with Piermarco:
1990 or 1992 (?): Vorau near Graz in Austria
with Professor Günter Leugering also

Joint works:

- Determination of source terms in a degenerate parabolic equation, Inverse Problems 26 (2010), 105003 (with Jacques Tort)
- Unique continuation and approximate controllability for a degenerate parabolic equation, Appl. Anal. 91 (2012), 1409-1425 (with Jacques Tort)
- Inverse source problem and null controllability for multidimensional parabolic operators of Grushin type. Inverse Problems 30 (2014), 025006 (with Karine Beauchard)
- Source reconstruction by partial measurements for a class of hyperbolic systems in cascade, "Mathematical Paradigms of Climate Science", Springer-INdAM Series vol. 15, 2016, pp.35-50 (with Fatiha Alabau-Boussouira)
- Inverse source problems with partial measurements for hyperbolic systems: uniqueness, non-uniqueness and stability (with Fatiha Alabau-Boussouira): ongoing
- Determination of moving sources (with Giuseppe Florindia): ongoing
Contents

- Part I. Inverse source problems
- Part II. Inverse problems for linear viscoelasticity
Part I. Inverse source problems

Simple inverse problem

\[
\begin{align*}
\partial_t u &= \Delta u + p(t)f(x), \quad x \in \Omega, \ 0 < t < T, \\
u|_{\partial \Omega} &= 0, \quad u(\cdot, 0) = 0.
\end{align*}
\]

IP 1: Given \( f(x) \), \( u(x_0, t) \), \( 0 < t < T \) \( \implies p(t) \)

IP 2: Given \( p(t) \), \( \partial_{\nu} u|_{\partial \Omega \times (0, T)} \) \( \implies f(x) \), \( x \in \Omega \)
IP 1.

\[ u(x, t) = \sum_{n=1}^{\infty} (f, \varphi_n) \varphi_n(x) \int_0^t e^{-\lambda_n(t-s)} p(s) ds \]

where \( \Delta \varphi_n = -\lambda_n \varphi_n \) and \( \varphi_n|_{\partial \Omega} = 0 \).

Set

\[ H(t) := \sum_{n=1}^{\infty} (f, \varphi_n) \varphi_n(x_0) e^{-\lambda_n t} \]

Then

\[ u(x_0, t) = \int_0^t H(t-s)p(s) ds, \quad 0 < t < T \]

If \( H(0) = f(x_0) \neq 0 \), then time-differentiation yields Volterra equation of second kind: \( \rightarrow \)

Inverse source problem is well-posed!

Uniqueness if \( f(x_0) = 0 \)
Some practical re-formulations make inverse source problems more complicated!
- No data from the explosion time
- Observation is started after explosion
- Estimation of total amount of emission
Oh, my god!
No data from explosion time!

\[ p(t) \]

\[ t=0: \text{explosion} \]

Observation starts
Reformulation I.
No date in (0, T)!
We have to determine which explosion happened in the past \(\Rightarrow\) archaeological inverse problem.

\[
\begin{aligned}
\partial_t u &= \Delta u + p(t)f(x), \quad x \in \Omega, \quad 0 < t < T, \\
u|\partial\Omega &= 0, \quad u(\cdot, 0) = 0.
\end{aligned}
\]

Let \(\text{supp } p \subset (0, \theta)\).

Determine \(p(t), \quad 0 < t < \theta\) by \(u(x_0, t)\),
Here \(\theta < t_0 < t < t_1\)
Conclusion with Jin Cheng (Fudan Univ.)

- Spatial one-dimensional: non-uniqueness
- Spatial-dimension $\geq 2$: uniqueness

\[ u(x_0, t) = \sum_{n=1}^{\infty} (f, \varphi_n) \varphi_n(x_0) e^{-\lambda_n t} \int_0^\theta e^{\lambda_n s} p(s) ds, \quad t > t_0 > \theta \]

Let $ (f, \varphi_n) \varphi_n(x_0) \neq 0, n \in \mathbb{N}$. Then

\[ \int_0^\theta e^{\lambda_n s} p(s) ds = 0, \quad n \in \mathbb{N} \]

Müntz theorem + asymptotics of $\lambda_n \implies$ Conclusion

Similar for wave equation
Reformulation II: Catch noisy guys!
with P. Cannarsa and G. Floridia

\[ \partial_t^2 u = \Delta u + f(x - \alpha(t)), \quad x \in \Omega, \ 0 < t < T, \]
\[ u|_{\partial \Omega} = 0, \quad u(\cdot, 0) = \partial_t u(\cdot, 0) = 0. \]

- Inverse moving source problem I:
  \( \alpha \): given, \( \omega \subset \Omega \),
  Determine \( f(x) \) by \( u|_{\omega \times (0,T)} \)

- Inverse moving source problem II:
  \( f \): given,
  Determine \( \alpha(t) \) by \( u(x_j, t), \ j = 1,\ldots, m, \ 0 < t < T \)
Part II. Inverse problems for linear viscoelasticity

Joint work with
Professor O. Imanuvilov (Colorado State University)
Main messages

With data on sub-boundary,
Carleman estimate for viscoelasticity equations yields

- Partial observability inequality
- Lipschitz stability for inverse problems of spatial varying Lamé coefficients
§1. Carleman estimate and inverse problem

Transfer argument to inverse problems by Carleman estimate

Source paper:

Modification by Imanuvilov and Yamamoto (2001)
Which is Carleman estimate?

Let $P$: differential operator. $\exists C > 0$ such that

$$\int_Q s|u|^2 e^{2s\varphi} \, dxdt \leq C \int_Q |Pu|^2 e^{2s\varphi} \, dxdt$$

for $\forall u \in C_0^\infty(Q)$ and all large $s > 0$.

- $\varphi(x, t)$: weight: choice is important.
- $C > 0$ is uniform in $s > 0$
How to get Carleman estimate

- "beautiful” general theory
- Hand-made way: Integration by Parts!!
  valid to *almost* all equations:

  - elliptic, parabolic, hyperbolic, Schrödinger, plate, Maxwell, Navier-Stokes, Lamé,
  - viscoelasticity, ultrahyperbolic, multi-time Schrödinger, first-order system,
  - parabolic-hyperbolic, conservation law-parabolic Lamé-parabolic, etc.

"General theory” does not work as prêt-à-porter for:
- many systems: Navier-Stokes equations, Lamé system
- plate equations
How to get stability for inverse problems?

Carleman estimate
+ transfer argument by Bukhgeim-Klibanov (1981), Klibanov (1992):
direct (elementary) calculus argument.
Essence of transfer argument

$\Omega \subset \mathbb{R}^n$: bounded domain

Inverse source problem:

Let $\omega \subset \Omega$, $0 < t_0 < T$. Determine $f$ in $\Omega$ by

\[ a := y(\cdot, t_0) \text{ and } y|_{\omega \times (0,T)} \text{ in} \]

\[
\left\{
\begin{array}{l}
\partial_t y = \Delta y + R(x, t)f(x) \quad \text{in } \Omega \times (0, T), \\
y|_{\partial \Omega} = 0.
\end{array}
\right.
\]
Inverse source problem: Let $\omega \subset \Omega$, $0 < t_0 < T$. Determine $f$ in $\Omega$ by $a := y(\cdot, t_0)$ and $y|_{\omega \times (0,T)}$ in $\partial_t y = \Delta y + R(x, t)f(x)$ in $\Omega_T := \Omega \times (0, T)$ with $y|_{\partial \Omega} = 0$.

Let $\alpha(\cdot, 0) = -\infty$ and $\alpha(\cdot, t_0) > \alpha(\cdot, t)$ for $t \neq t_0$.

Set $y_k = \partial_t^k y$, $k = 1, 2 \implies \partial_t y_k = \Delta y_k + \partial_t^k R(x, t)f(x)$, $y_k|_{\partial \Omega} = 0$.

$\partial_t y(x, t_0) = \Delta a + R(x, t_0)f(x)$

Carleman estimate: $\int_{\Omega_T} s|\partial_t^k y|^2 e^{2s\alpha} \, dx \, dt \leq C \int_{\Omega_T} |f|^2 e^{2s\alpha} \, dx \, dt + C\|\text{Data}\|_\ast^2$

$\alpha(\cdot, 0) = -\infty$ and $\alpha(\cdot, t_0) \geq \alpha(\cdot, t) \implies \int_{\Omega} |\partial_t y(x, t_0)|^2 e^{2s\alpha(x,t_0)} \, dx$

$$= \int_0^{t_0} \partial_t \left( \int_{\Omega} |\partial_t y|^2 e^{2s\alpha} \, dx \right) \, dt \leq C \int_{\Omega_T} (|\partial_t^2 y\partial_t y| + s|\partial_t \alpha| |\partial_t y|^2) e^{2s\alpha} \, dx \, dt$$

$$\leq C \int_{\Omega_T} |f|^2 e^{2s\alpha} \, dx \, dt + C\|\text{Data}\|_\ast^2 \quad \text{(by Carleman estimate)}$$

$$= C \int_{\Omega} |f|^2 e^{2s\alpha(x,t_0)} \left( \int_0^T e^{2s(\alpha(x,t)-\alpha(x,t_0))} \, dt \right) \, dx + C\|\text{Data}\|_\ast^2$$
How can we describe main results by Carleman estimates?

- Usually **single measurement**: extra data for one initial-boundary value problem
- Lipschitz stability (usually)
- **Drawback**: positivity (non-degeneracy) condition on initial data
Comments.

- the best possible stability
- positivity of initial displacement

Justification of the positivity: finite number of repeats of measurements may cover positivity!

Seismological inverse problem: one earthquake $\implies$ one initial-boundary value problem $u(\cdot, 0)$ and $u|_{\omega \times (0, T)}$: recorded by each earthquake

Very frequent earthquake (e.g., in Japan "fortunately") $\implies$

In data set ${a_k, u(a_k)|_{\omega \times (0, T)}}_{k=1,2,...,N}$, we can eventually obtain $\bigcup_k \text{supp } a_k \supset \Omega$!

Hit objective many times!
Supplement: Extension of research scopes

\[
\partial_t u + q(x, t)\partial_t^\beta u = \Delta u + A(x, t) \cdot \nabla u + C(x, t)u
\]

in \( \Omega \times (0, T) \)

Set \( \partial_t^\beta u(x, t) = \frac{1}{\Gamma(1-\beta)} \int_0^t (t - \eta)^{-\beta} \partial_t^\eta u(x, \eta) d\eta, \ 0 < \eta < 1: \)

Caputo fractional derivative

\( \iff \) Many applications: anomalous diffusion of Cs137, design of geothermal plant

Carleman estimate established: \( \beta < \frac{1}{2} \) or \( \beta \leq \frac{3}{4}, \in Q \)

(Z.Li, X.Huang and Y.)
Incomplete references on Carleman estimate and related inverse problems for fractional partial differential equations

- Xu-Cheng-Yamamoto (2011)
- Yamamoto-Zheng
- Z.Li-Imanuvilov-Yamamoto: DN map (2016)
- Kian-Oksanen-Soccorsi-Yamamoto (DN map at one shot)

Inverse problems for fractional differential equations: rapidly developing
§2. Linear viscoelasticity

§1. What is viscoelasticity?

- viscosity ← fluid
- elasticity ← solid

Viscoelasticity: exhibiting viscosity and elasticity

Example: many materials
- honey, gum, white of egg
- polymers, semiconductor thin film
- human body ⇒ Medical purpose
Viscoelastic Model

- Spring models elasticity
- Dashpot models viscosity
§3. Formulation and main results I

\[ \Omega \subset \mathbb{R}^3: \text{bounded domain} \]

\[ \rho(x)\partial_t^2 u = L_{\lambda, \mu}(x)u - \int_0^t L_{\lambda, \mu}(x, t, \eta)u(x, \eta)d\eta + F, \]

in \( Q := \Omega \times (-T, T) \),

\[ u|_{\partial\Omega} = 0, \quad \partial^j_t u(\cdot, \pm T) = 0, \quad j = 0, 1. \]

Let \( u = (u_1, u_2, u_3)^T \), \( \nabla = (\partial_1, \partial_2, \partial_3) \),

\[ L_{\lambda, \mu}(x, t, \eta)u = \tilde{\mu}(x, t, \eta)\Delta u + (\tilde{\mu} + \tilde{\lambda}(x, t, \eta))\nabla \text{div } u \]

\[ + (\text{div } u)\nabla \tilde{\lambda} + ((\nabla u) + (\nabla u)^T)\nabla \tilde{\mu} \]

- Carleman estimate
- global Lipschitz stability for inverse problems
Let \( \rho, \mu, \lambda \in C^2(\Omega), \rho, \mu, \lambda + \mu > 0, \)
\( \tilde{\lambda}, \tilde{\mu} \in C^2(\Omega \times ([-T, 0] \cup [0, T])^2), \)
Set \( p_\beta(x, \xi) := \rho(x)\xi_0^2 - \beta(x)|\xi'|^2 \) for \( \xi = (\xi_0, \xi_1, \xi_2, \xi_3) =: (\xi_0, \xi') \).
Assume:
- \( \psi(x, t) := d(x) - \gamma t^2 \) is pseudoconvex with respect to \( p_\mu \) and \( p_{\lambda+2\mu} \).
- \( (\nabla d(x) \cdot \nu(x)) \geq 0 \) for \( x \in \Gamma \subset \partial \Omega \).
\( \nu(x) \): unit outward normal vector to \( \partial \Omega \).
Let $t := x_0$, $\xi = (\xi_0, \xi_1, \xi_2, \xi_3) = (\xi_0, \xi')$, and $\{ \varphi, \psi \} = \sum_{j=0}^{3} \frac{\partial \varphi}{\partial \xi_j} \frac{\partial \psi}{\partial x_j} - \frac{\partial \psi}{\partial \xi_j} \frac{\partial \varphi}{\partial x_j}$,

\[ p_\beta(x, \xi) = \rho(x)\xi_0^2 - \beta(x)|\xi'|^2, \quad \psi(x, t) = d(x) - \tau t^2. \]

Pseudoconvex:

- $\psi \in C^3$, $\nabla d \neq 0$ on $\overline{\Omega}$,

\[
\{ p_\beta, \{ p_\beta, \psi \} \}(x, t, \xi) > 0, \quad \beta \in \{ \mu, \lambda + 2\mu \}
\]

for $(x, t, \xi) \in \overline{Q} \times (\mathbb{R}^4 \setminus \{ 0 \})$ satisfying $p_\beta(x, \xi) = (\partial_{\xi_0} p_\beta)(\partial_t \psi) + \langle \nabla p_\beta, \nabla \psi \rangle = 0$.

- \[
\frac{1}{2 \sqrt{-1s}} \left\{ p_\beta(x, \xi - \sqrt{-1s} \nabla_{x,t} \psi), p_\beta(x, \xi + \sqrt{-1s} \nabla_{x,t} \psi(x, t)) \right\} > 0, \quad \beta \in \{ \mu, \lambda + 2\mu \}
\]

if $s > 0$ and $(x, t, \xi) \in \overline{Q} \times (\mathbb{R}^4 \setminus \{ 0 \})$ satisfying

\[
p_\beta(x, \xi + \sqrt{-1s} \nabla_{x,t} \psi)
\]

\[
= \partial_{\xi_0} p_\beta(x, \xi + \sqrt{-1s} \nabla_{x,t} \psi) \partial_t \psi + \nabla_\xi p_\beta(x, \xi + \sqrt{-1s} \nabla_{x,t} \psi), \nabla \psi \rangle = 0.
\]
Observability inequality:

\[
\rho(x) \partial_t^2 u(x, t) = L_{\lambda, \mu}(x) u(x, t) - \int_0^t L_{\lambda, \mu}^e(x, t, \eta) u(x, \eta) d\eta, \quad \text{in } \Omega \times (0, T),
\]

\[
u|_{\partial \Omega} = 0,
\]

\[
u(\cdot, 0) = 0, \quad \partial_t u(\cdot, 0) = a.
\]

**Theorem 1 (observability inequality).**
Let \( \Gamma \supset \{ x \in \partial \Omega; (\nabla d(x) \cdot \nu(x)) \geq 0 \} \) and \( T > T_0(\rho, \lambda, \mu, \Omega, \psi) : \text{large} \). Then

\[
\| \partial_t u(\cdot, 0) \|_{H^1(\Omega)} \leq C(\| \nabla_{x,t} \nu u \|_{L^2(0,T; L^2(\Gamma))} + \| \partial^2_{\nu} u \|_{L^2(0,T; L^2(\Gamma))}).
\]

Proof is routine by the Carleman estimate.
\[
\begin{aligned}
\rho(x) \frac{\partial^2 y(x, t)}{\partial t^2} &= L_{\lambda, \mu}(x) y - \int_0^t L_{\lambda, \mu}(x, t, \eta) y(x, \eta) \, d\eta + R(x, t) f(x) \\
\text{in } &\Omega \times (0, T), \\
y|_{\partial \Omega} = 0, &\quad y(\cdot, 0) = \partial_t y(\cdot, 0) = 0.
\end{aligned}
\]

\(R(x, t): 3 \times 3, f: \mathbb{R}^3\)-valued

Inverse source problem. \(R\): given, \(\Gamma \supset \{x \in \partial \Omega; (\nabla d(x) \cdot \nu(x)) \geq 0\}\)
Determine \(f\) in \(\Omega\) by \(\partial \nu y|_{\Gamma \times (0, T)}\).

Theorem 2 (inverse source problem).
Let \(\det |R(\cdot, 0)| \neq 0\) and \(\partial_t R(\cdot, 0) = 0\) on \(\overline{\Omega}\), \(T > T_0(\rho, \lambda, \mu, \Omega, \psi):\) large. Then

\[
\|f\|_{H^1(\Omega)} \leq C(\|\nabla_{x,t} \partial \nu y\|_{H^1(0, T; L^2(\Gamma))} + \|\partial^2_{\nu} y\|_{H^1(0, T; L^2(\Gamma))})
\]

for \(f \in H^1_0(\Omega)\).
\[\Box \rho, \beta := \rho(x) \partial_t^2 - \beta(x) \Delta\]

\[
\begin{align*}
\rho(x) \partial_t^2 y - L_{\lambda, \mu} y + \int_0^t L_{\lambda, \mu} y(x, \eta) d\eta &= F, \\
\partial_t^j y(\cdot, \pm T) &= 0, \quad j = 0, 1.
\end{align*}
\]

Let

\[
\begin{align*}
v_1 &:= \mathsf{rot} y - \int_0^t \frac{\tilde{\mu}(x, t, \eta)}{\mu(x)} \mathsf{rot} y(x, \eta) d\eta, \\
v_2 &:= \mathsf{div} y - \int_0^t \frac{(\lambda + 2\mu)(x, t, \eta)}{(\lambda + 2\mu)(x)} \mathsf{div} y(x, \eta) d\eta,
\end{align*}
\]

\[\Rightarrow\]

\[
\begin{align*}
\Box \rho, \mu v_1 &= [\text{lower order terms of } \mathsf{grad} x, t v_1, \mathsf{grad} x, t v_2] + \int_0^t [\Delta v_1, \Delta v_2] d\eta, \\
\Box \rho, \lambda + 2\mu v_2 &= [\text{lower order terms of } \mathsf{grad} x, t v_1, \mathsf{grad} x, t v_2] + \int_0^t [\Delta v_1, \Delta v_2] d\eta, \\
\text{Boundary condition} : \mathcal{B}(v_1, v_2) &= g
\end{align*}
\]
Theorem 3 (key Carleman estimate).

Let $Q := \Omega \times (-T, T)$, $v := (v_1, v_2) \in H^1(Q)$,

Let $\varphi(x, t) := e^{\tau \psi}$, $\tau > 0$: large, $\psi(x, t) = d(x) - \beta t^2$, $\beta > 0$: small

\[
\int_Q (s|\nabla v|^2 + s^3 |v|^2)e^{2s\varphi} \, dxdt \\
\leq C \int_Q (|\text{rot } F|^2 + |\text{div } F|^2 + |F|^2)e^{2s\varphi} \, dxdt + Ce^{Cs} (\|\nabla_{x,t} \varphi \|^2_{L^2(\Gamma \times (-T,T))} + \|\partial^2_y \varphi \|_{L^2}).
\]

for all large $s > 0$.

Here $\rho(x) \partial_t^2 y - L_{\lambda, \mu} y + \int_0^t L_{\lambda, \mu} \tilde{y}(x, \eta)d\eta = F$ and $\partial_j^i y(\cdot, \pm T) = 0$ with $j = 0, 1$.

\[
\begin{aligned}
    v_1 &:= \text{rot } y - \int_0^t \frac{\tilde{\mu}(x,t,\eta)}{\mu(x)} \text{rot } y(x, \eta)d\eta, \\
v_2 &:= \text{div } y - \int_0^t \frac{(\lambda + 2\tilde{\mu})(x,t,\eta)}{(\lambda + 2\mu)(x)} \text{div } y(x, \eta)d\eta,
\end{aligned}
\]
Our Carleman estimate does not assume compact support for \( y \). \( \Rightarrow \) \textbf{Lipschitz stability:}

\[
\|\text{coefficient}\| \leq C \|\text{Data}\|_{\ast}(\partial \Omega \times (0,T))
\]

With Carleman estimate for compact supports, we can get only \textbf{Hölder stability}

\textbf{Partial references on Hölder stability:}
1. de Buhan-Osses (2010)
2. Lorenzi-Messina-Romanov (2007),
   Lorenzi-Romanov (2006)
Recipe for Proof of Carleman estimate for Lamé system

- Decoupling
- Reduce integral term to lower term

No general theory for Carleman estimate for strongly coupled systems:

\[ \rho(x) \partial^2_t y = \mu(x) \Delta y + (\lambda + \mu) \nabla \text{div } y + \text{lower order terms} \]

Decoupling into weakly coupled system (by Fichera, etc.):
References.

Recipe for integral term

Cavaterra-Lorenzi-Yamamoto: 2006

\[ \partial_t^2 u = p(x)\Delta u + \int_0^t K(x, t, \eta)\Delta u(x, \eta)\,d\eta + F \]

Key invertible integral transform:

\[ v(x, t) = p(x)u(x, t) + \int_0^t K(x, t, \eta)u(x, \eta)\,d\eta \implies \]

\[ \partial_t^2 v = p(x)\Delta v + \mathcal{P}_1 u(x, t) + \int_0^t K(x, t, \eta)\mathcal{P}_2 u(x, \eta)\,d\eta = \text{lower term} + pF \]

Here \( \mathcal{P}_j \): first-order operators
Key Lemma (Klibanov):

$\varphi(x, t)$: maximum at $t = t_0$. Then

$$\int_Q \left| \int_{t_0}^t |w(x, \eta)| d\eta \right|^2 e^{2s\alpha} dx dt \leq \frac{C}{s} \int_Q |w|^2 e^{2s\varphi(x,t)} dx dt.$$

This lemma + hyperbolic Carleman estimate $\implies$
Theorem 0.
Let $u(\cdot, 0) = 0$ or $\partial_t u(\cdot, 0) = K(\cdot, 0, 0) = 0$. Then

$$\int_Q (s|\nabla_{x,t}u|^2 + s^3|u|^2)e^{2s\alpha} \, dxdt \leq C \int_Q |F|^2 e^{2s\alpha} \, dxdt$$

for all large $s > 0$ and $\text{supp } u \subset \Omega \times (0, T)$ satisfying

$$\partial_t^2 u = p(x)\Delta u + \int_0^t K(x, t, \eta)\Delta u(x, \eta) \, d\eta + F$$
§5. Determination of viscoelastic coefficients

\[ \begin{cases} \partial_t^2 u_k = L_{\lambda,\mu}(x)u_k + \int_0^t L_{\tilde{\lambda}_k,\tilde{\mu}_k}(x, t, \eta)u_k(x, \eta)d\eta, \\ \text{in } \Omega \times (-T, T), \\ u_k|_{\partial\Omega} : \text{given}, \\ u_k(\cdot, 0) = a, \quad \partial_t u_k(\cdot, 0) = 0, \quad k = 1, 2. \end{cases} \]

Here \( \tilde{\lambda}_k(x, t, \eta) = \ell_k(x)p(t, \eta), \tilde{\mu}_k(x, t, \eta) = m_k(x)q(t, \eta). \)

\( p, q: \text{known}, \quad p(0, 0), q(0, 0) \neq 0. \)

Coefficient inverse problem with two measurements.

\[ ||\ell_1 - \ell_2|| + ||m_1 - m_2|| \leq C \sum_{i=1}^2 (||\partial_\nu(u_1(a_i) - u_2(a_i))||?) \]
Coefficient inverse problem $\Rightarrow y := u_1 - u_2 \Rightarrow$
Inverse source problem

Non-degeneracy condition similar to $\det R(\cdot, 0) \neq 0$ for inverse source problem requires conditions for $a_1, a_2$. 
Theorem 4.
Assume $\ell_k, m_k \in C^4(\Omega)$, $\ell_k, m_k, \partial_\nu \ell_k, \partial_\nu m_k$: given on $\partial \Omega$, $\|u_k\|_{W^{8,\infty}(\Omega \times (-T,T))} \leq M$: a priori bound.

Let

- $T > T_0$ (large constant).
- Let a pair of initial displacement $a_1, a_2$ satisfy non-degeneracy condition (*)

$$\Rightarrow \quad \|\ell_1 - \ell_2\|_{H^1(\Omega)} + \|m_1 - m_2\|_{H^1(\Omega)}$$

$$\leq C \sum_{i=1}^{2} (\|\nabla_{x,t} \partial_\nu (u_1(a_i) - u_2(a_i))\|_{H^2(0,T;L^2(\Gamma))} + \|\partial_\nu^2 (u_1(a_i) - u_2(a_i))\|_{H^2(L^2)})$$
Non-degeneracy condition (*) for initial data $a_1, a_2$

For $6 \times 4$ matrix $A$, we set
\[ \{A\}_{i,j} : \text{matrix created by deleting } i\text{-th and } j\text{-th rows of } A, \]
\[ \det_{i,j} A = \det \{A\}_{i,j} \]
$E_3 : 3 \times 3$ identity matrix.

Non-degeneracy condition (*) $\iff \exists i_1, i_2, j_1, j_2 \in \{1, 2, ..., 6\}$ such that
\[
\det_{i_1,j_2} \begin{pmatrix}
(div a_1)E_3 & (\nabla a_1 + (\nabla a_1)^T)\nabla d \\
(div a_2)E_3 & (\nabla a_2 + (\nabla a_2)^T)\nabla d 
\end{pmatrix} \neq 0,
\]
\[
\det_{j_1,j_2} \begin{pmatrix}
\nabla a_1 + (\nabla a_1)^T & (\nabla a_1)^T \nabla d \\
\nabla a_2 + (\nabla a_2)^T & (\nabla a_2)^T \nabla d 
\end{pmatrix} \neq 0
\]
on $\Omega$. 
Example of $a_1, a_2$. Let $\Omega \subset \{(x_1, x_2, x_3); x_1, x_3 > 0\}$, $d(x) = |x|^2,$

$$p(t, \eta) = q(t, \eta) =: \overline{p}(t - \eta), \quad \overline{p}(0) = 1.$$  

Then $a_1 = \begin{pmatrix} 0 & x_1x_2 \\ x_1x_2 & 0 \end{pmatrix}$ and $a_2 = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$ satisfy (*):

$$(*) \implies \begin{vmatrix} x_1 & 0 & 0 & 2x_2^2 \\ 0 & x_1 & 0 & 6x_1x_2 \\ 0 & 0 & x_1 & 0 \\ 2 & 0 & 0 & 4x_1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 4x_3 \end{vmatrix} = -8x_1^2x_3 \neq 0 \quad \text{on } \overline{\Omega}.$$  

$$\begin{vmatrix} 0 & x_2 & 0 & 2x_1^2 \\ x_2 & 2x_1 & 0 & 2x_1x_2 \\ 0 & 0 & 0 & 2x_1x_3 \\ 2 & 0 & 0 & 4x_1 \\ 0 & 0 & 0 & 4x_2 \\ 0 & 0 & 2 & 4x_3 \end{vmatrix} = 16x_1^3 \neq 0 \quad \text{on } \overline{\Omega}.$$
Main messages

With data on sub-boundary, Carleman estimate for viscoelasticity equations yields

- Partial observability inequality
- Lipschitz stability for inverse problems of spatial varying Lamé coefficients
Age 60 = one cycle of the life calendar

Life Begins At Sixty!
PIERMARCO,
CONGRATULATIONS
ON THE 60TH BIRTHDAY
GOOD HEALTH!

KANREKI
祝