Carleman estimates for viscoelasticity equations and applications

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New Trends in Control Theory and PDEs

on the occasion of the 60th birthday of Professor Piermarco Cannarsa

INdAM - Rome 7 July 2017

First meeting (1990?) at Vorau (Steiermark)

First meeting with Piermarco:
1990 or 1992 (?): Vorau near Graz in Austria
with Professor Günter Leugering also

Joint works:

- Determination of source terms in a degenerate parabolic equation, Inverse Problems **26** (2010), 105003 (with Jacques Tort)
- Unique continuation and approximate controllability for a degenerate parabolic equation, Appl. Anal. **91** (2012), 1409-1425 (with Jacques Tort)
- Inverse source problem and null controllability for multidimensional parabolic operators of Grushin type. Inverse Problems **30** (2014), 025006 (with Karine Beauchard)
- Source reconstruction by partial measurements for a class of hyperbolic systems in cascade, ”Mathematical Paradigms of Climate Science” , Springer-INdAM Series vol. 15, 2016, pp.35-50 (with Fatiha Alabau-Boussouira)
- Inverse source problems with partial measurements for hyperbolic systems: uniqueness, non-uniqueness and stability (with Fatiha Alabau-Boussouira): ongoing
- Determination of moving sources (with Giuseppe Florindia): ongoing

Contents

- Part I. Inverse source problems
- Part II. Inverse problems for linear viscoelasticity

Part I. Inverse source problems

Simple inverse problem

$$\begin{cases} \partial_t u = \Delta u + p(t) f(x), & x \in \Omega,\ 0 < t < T, \\ u|_{\partial\Omega} = 0, \quad u(\cdot, 0) = 0. \end{cases}$$

IP 1: Given $f(x)$, $u(x_0, t)$, $0 < t < T \Longrightarrow p(t)$

IP 2: Given $p(t)$, $\partial_\nu u|_{\partial\Omega\times(0,T)} \Longrightarrow f(x)$, $x \in \Omega$

IP 1.

$$u(x,t) = \sum_{n=1}^{\infty} (f, \varphi_n)\varphi_n(x) \int_0^t e^{-\lambda_n (t-s)} p(s)ds$$

where $\Delta\varphi_n = -\lambda_n \varphi_n$, $\varphi_n|_{\partial\Omega} = 0$.
Set

$$H(t) := \sum_{n=1}^{\infty} (f, \varphi_n)\varphi_n(x_0) e^{-\lambda_n t}$$

Then

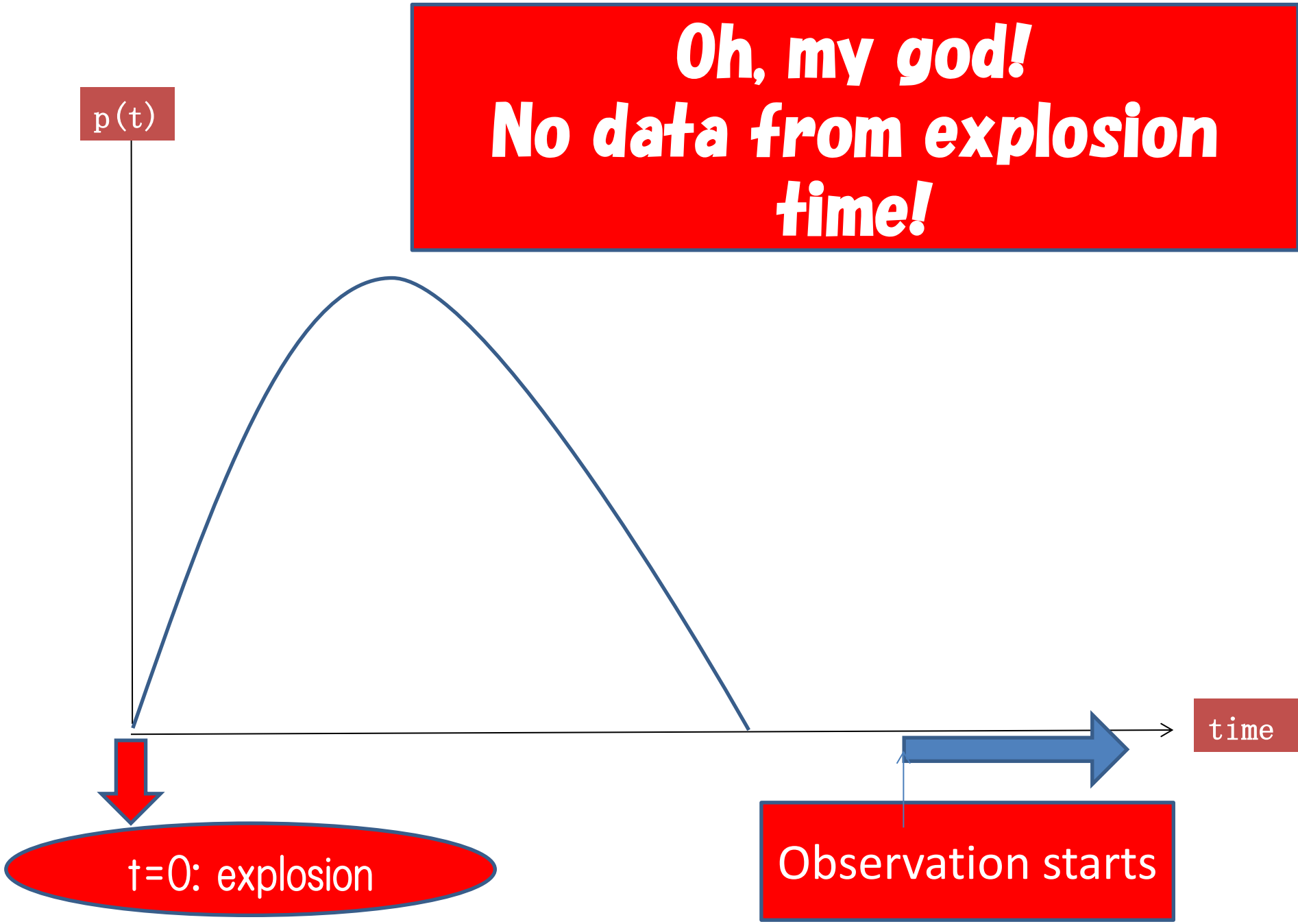
$$u(x_0, t) = \int_0^t H(t-s)p(s)ds, \quad 0 < t < T$$

If $H(0) = f(x_0) \neq 0$, then time-differentiation yields Volterra equation of second kind: $\longrightarrow$
Inverse source problem is well-posed!

Uniqueness if $f(x_0) = 0$

Some practical re-formulations make inverse source problems more complicated!

- No data from the explosion time
- Observation is started after explosion
- Estimation of total amount of emission


Oh, my god!
No data from explosion time!
p(t)
time
t=0: explosion
Observation starts

Reformulation I.

No date in $(0, T)$! $\Longrightarrow$

We have to determine which explosion happened in the past $\Longrightarrow$ archaeological inverse problem.

$$\begin{cases} \partial_t u = \Delta u + p(t) f(x), & x \in \Omega,\ 0 < t < T, \\ u|_{\partial\Omega} = 0, \quad u(\cdot, 0) = 0. \end{cases}$$

Let supp $p \subset (0, \theta)$.

Determine $p(t)$, $0 < t < \theta$ by $u(x_0, t)$,

Here $\theta < t_0 < t < t_1$

Conclusion with Jin Cheng (Fudan Univ.)

- Spatial one-dimensional: non-uniqueness
- Spatial-dimension ≥ 2: uniqueness

$$u(x_0, t) = \sum_{n=1}^{\infty} (f, \varphi_n)\varphi_n(x_0) e^{-\lambda_n t} \int_0^{\theta} e^{\lambda_n s} p(s) ds, \quad t > t_0 > \theta$$

Let $(f, \varphi_n)\varphi_n(x_0) \neq 0$, $n \in \mathbf{N}$. Then

$$\int_0^{\theta} e^{\lambda_n s} p(s) ds = 0, \quad n \in \mathbf{N}$$

Müntz theorem + asymptotics of $\lambda_n \Longrightarrow$ Conclusion

Similar for wave equation

Reformulation II: Catch noisy guys!

with P. Cannarsa and G. Floridia

$$\begin{cases} \partial_t^2 u = \Delta u + f(x - \alpha(t)), & x \in \Omega,\ 0 < t < T, \\ u|_{\partial\Omega} = 0, \quad u(\cdot, 0) = \partial_t u(\cdot, 0) = 0. \end{cases}$$

- Inverse moving source problem I:

  α: given, $\omega \subset \Omega$,

  Determine $f(x)$ by $u|_{\omega \times (0,T)}$
- Inverse moving source problem II:

  f: given,

  Determine $\alpha(t)$ by $u(x_j, t)$, $j = 1, ..., m$, $0 < t < T$

Part II. Inverse problems for linear viscoelasticity

Joint work with

Professor O. Imanuvilov (Colorado State University)

Main messages

With data on sub-boundary,

Carleman estimate for viscoelasticity equations yields

- Partial observability inequality
- Lipschitz stability for inverse problems of spatial varying Lamé coefficients

§1. Carleman estimate and inverse problem

Transfer argument to inverse problems by Carleman estimate

Source paper:

Klibanov (1991), Bukhgeim and Klibanov (1981)

Modification by Imanuvilov and Yamamoto (2001)

Which is Carleman estimate?

Let P: differential operator. $\exists C > 0$ such that

$$\int_Q s|u|^2 e^{2s\varphi}\,dxdt \leq C \int_Q |Pu|^2 e^{2s\varphi}\,dxdt$$

for $\forall u \in C_0^\infty(Q)$ and all large $s > 0$.

- $\varphi(x, t)$: weight: choice is important.
- $C > 0$ is uniform in $s > 0$

How to get Carleman estimate

- ”beautiful” general theory
- Hand-made way: **Integration by Parts!!**

  valid to almost all equations:

  elliptic, parabolic, hyperbolic, Schrödinger, plate, Maxwell, Navier-Stokes, Lamé,

  viscoelasticity, ultrahyperbolic, multi-time Schrödinger, first-order system,

  parabolic-hyperbolic, conservation law-parabolic Lamé-parabolic, etc.

”General theory” does not work as prêt-à-porter for:

- many systems: Navier-Stokes equations, Lamé system
- plate equations

How to get stability for inverse problems?

Carleman estimate

+ transfer argument by Bukhgeim-Klibanov (1981),

Klibanov (1992):

direct (elementary) calculus argument.

Essence of transfer argument

$\Omega \subset \mathbf{R}^n$: bounded domain

Inverse source problem:

Let $\omega \subset \Omega$, $0 < t_0 < T$. Determine f in Ω by $a := y(\cdot, t_0)$ and $y|_{\omega \times (0,T)}$ in

$$\begin{cases} \partial_t y = \Delta y + R(x,t) f(x) & \text{in } \Omega \times (0,T), \\ y|_{\partial\Omega} = 0. \end{cases}$$

Inverse source problem: Let $\omega \subset \Omega$, $0 < t_0 < T$. Determine f in Ω by $a := y(\cdot, t_0)$ and $y|_{\omega\times(0,T)}$ in $\partial_t y = \Delta y + R(x,t)f(x)$ in $\Omega_T := \Omega \times (0,T)$ with $y|_{\partial\Omega} = 0$.

Let $\alpha(\cdot, 0) = -\infty$ and $\alpha(\cdot, t_0) > \alpha(\cdot, t)$ for $t \neq t_0$.

Set $y_k = \partial_t^k y$, $k = 1, 2 \Longrightarrow \partial_t y_k = \Delta y_k + \partial_t^k R(x,t)f(x)$, $y_k|_{\partial\Omega} = 0$.

$\partial_t y(x, t_0) = \Delta a + R(x, t_0)f(x)$

Carleman estimate: $\int_{\Omega_T} s|\partial_t^k y|^2 e^{2s\alpha}\,dxdt \le C \int_{\Omega_T} |f|^2 e^{2s\alpha}\,dxdt + C||\text{Data}||_*^2$

$\alpha(\cdot, 0) = -\infty$ and $\alpha(\cdot, t_0) \ge \alpha(\cdot, t) \Longrightarrow \int_\Omega |\partial_t y(x, t_0)|^2 e^{2s\alpha(x,t_0)}\,dx$

$$= \int_0^{t_0} \partial_t \left(\int_\Omega |\partial_t y|^2 e^{2s\alpha}\,dx \right) dt \le C \int_{\Omega_T} (|\partial_t^2 y \partial_t y| + s|\partial_t \alpha||\partial_t y|^2) e^{2s\alpha}\,dxdt$$

$$\le C \int_{\Omega_T} |f|^2 e^{2s\alpha}\,dxdt + C||\text{Data}||_*^2 \quad \text{(by Carleman estimate)}$$

$$= C \int_\Omega |f|^2 e^{2s\alpha(x,t_0)} \underbrace{\left(\int_0^T e^{2s(\alpha(x,t)-\alpha(x,t_0))}\,dt \right)}_{=o(1)} dx + C||\text{Data}||_*^2$$

How can we describe main results by Carleman estimates?

- Usually single measurement: extra data for one initial-boundary value problem
- Lipschitz stability (usually)
- Drawback: positivity (non-degeneracy) condition on initial data

Comments.

- the best possible stability
- positivity of initial displacement

Justification of the positivity: finite number of repeats of measurements may cover positivity!
Seismological inverse problem: one earthquake $\Longrightarrow$ one initial-boundary value problem
$u(\cdot, 0)$ and $u|_{\omega\times(0,T)}$: recorded by each earthquake
Very frequent earthquake (e.g., in Japan "fortunately")
$\Longrightarrow$

In data set $\{a_k, u(a_k)|_{\omega\times(0,T)}\}_{k=1,2,\ldots,N}$, we can eventually obtain $\bigcup_k \operatorname{supp} a_k \supset \Omega$!

Hit objective many times!

Supplement: Extension of research scopes

$$\partial_t u + q(x,t)\partial_t^\beta u = \Delta u + A(x,t)\cdot\nabla u + C(x,t)u$$

in $\Omega \times (0,T)$

Set $\partial_t^\beta u(x,t) = \frac{1}{\Gamma(1-\beta)}\int_0^t (t-\eta)^{-\beta}\partial_\eta u(x,\eta)d\eta$, $0 < \eta < 1$:

Caputo fractional derivative

$\Longleftarrow$ Many applications: anomalous diffusion of Cs137, design of geothermal plant

Carleman estimate established: $\beta < \frac{1}{2}$ or $\beta \leq \frac{3}{4}, \in \mathbb{Q}$

(Z.Li, X.Huang and Y.)

Incomplete references on Carleman estimate and related inverse problems for fractional partial differential equations

- Xu-Cheng-Yamamoto (2011)
- Yamamoto-Zheng
- J. Cheng-C.L. Lin-G. Nakamura (2013)
- Z.Li-Imanuvilov-Yamamoto: DN map (2016)
- Kian-Oksanen-Soccorsi-Yamamoto (DN map at one shot)

Inverse problems for fractional differential equations: rapidly developing

§2. Linear viscoelasticity

§1. What is viscoelasticity?

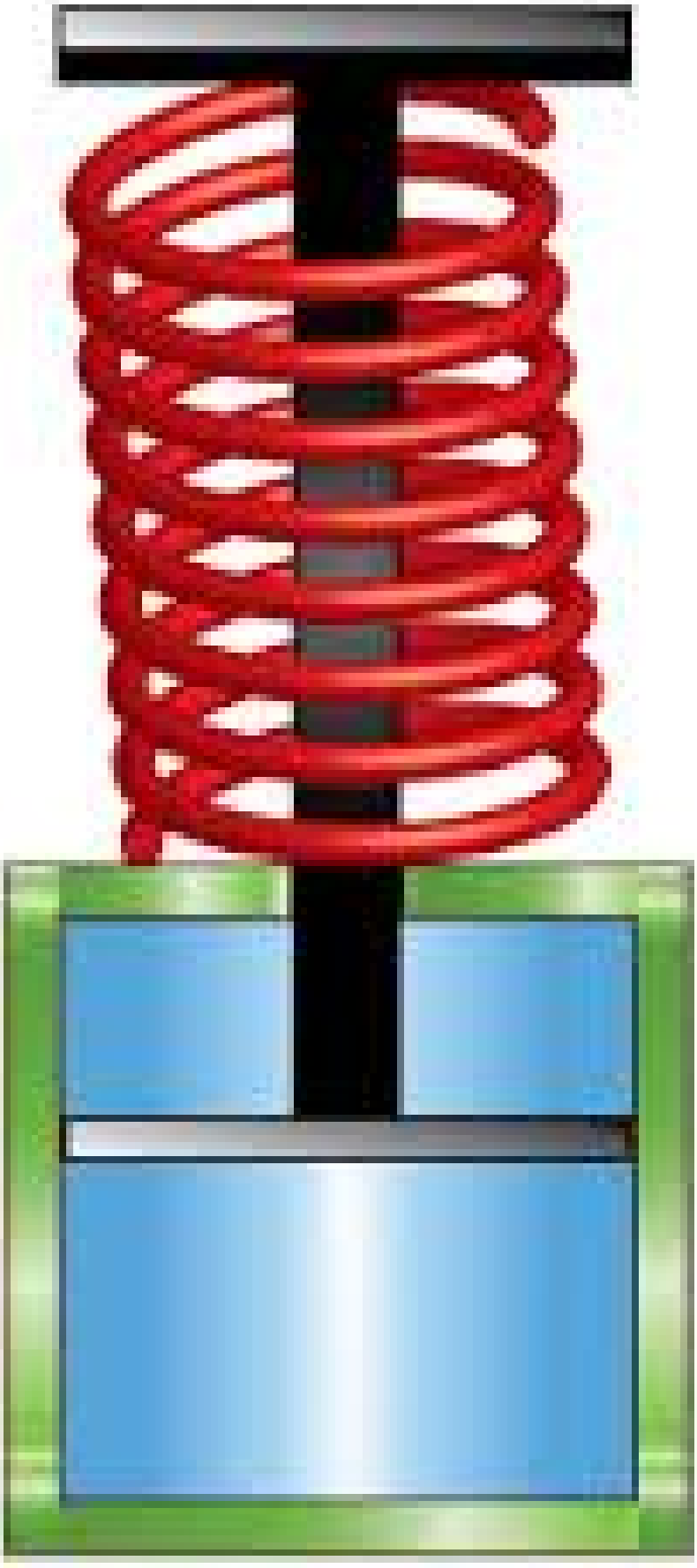
- viscosity $\longleftarrow$ fluid
- elasticiy $\longleftarrow$ solid

viscoelasticity: exhibiting viscosity and elasticity

Example: many materials

- honey, gum, white of egg
- polymers, semiconductor thin film
- human body $\Longrightarrow$ Medical purpose

Viscoelastic Model



Spring models elasticity

Dashpot models viscosity

§3. Formulation and main results I

$\Omega \subset \mathbf{R}^3$: bounded domain

$$\rho(x)\partial_t^2 u = L_{\lambda,\mu}(x)u - \int_0^t L_{\widetilde{\lambda},\widetilde{\mu}}(x,t,\eta)u(x,\eta)d\eta + F,$$

in $Q := \Omega \times (-T,T)$,

$$u|_{\partial\Omega} = 0, \quad \partial_t^j u(\cdot, \pm T) = 0, \ j = 0, 1.$$

Let $u = (u_1, u_2, u_3)^T$, $\nabla = (\partial_1, \partial_2, \partial_3)$,

$L_{\widetilde{\lambda},\widetilde{\mu}}(x,t,\eta)u = \widetilde{\mu}(x,t,\eta)\Delta u + (\widetilde{\mu} + \widetilde{\lambda}(x,t,\eta))\nabla \mathrm{div}\, u$

$+(\mathrm{div}\, u)\nabla\widetilde{\lambda} + ((\nabla u) + (\nabla u)^T)\nabla\widetilde{\mu} \Longrightarrow$

- Carleman estimate
- global Lipschitz stability for inverse problems

Let

$\rho, \mu, \lambda \in C^2(\overline{\Omega})$, $\rho, \mu, \lambda + \mu > 0$,

$\widetilde{\lambda}, \widetilde{\mu} \in C^2(\overline{\Omega} \times ([-T, 0] \cup [0, T])^2)$,

Set $p_\beta(x, \xi) := \rho(x)\xi_0^2 - \beta(x)|\xi'|^2$ for

$\xi = (\xi_0, \xi_1, \xi_2, \xi_3) =: (\xi_0, \xi')$.

Assume:

- $\psi(x, t) := d(x) - \gamma t^2$ is pseudoconvex with respect to p_μ and $p_{\lambda+2\mu}$.
- $(\nabla d(x) \cdot \nu(x)) \geq 0$ for $x \in \Gamma \subset \partial\Omega$.

$\nu(x)$: unit outward normal vector to $\partial\Omega$.

Let $t := x_0$, $\xi = (\xi_0, \xi_1, \xi_2, \xi_3) = (\xi_0, \xi')$, and $\{\varphi, \psi\} = \sum_{j=0}^{3} \frac{\partial \varphi}{\partial \xi_j} \frac{\partial \psi}{\partial x_j} - \frac{\partial \psi}{\partial \xi_j} \frac{\partial \varphi}{\partial x_j}$,

$p_\beta(x, \xi) = \rho(x)\xi_0^2 - \beta(x)|\xi'|^2$, $\psi(x, t) = d(x) - \tau t^2$.

Pseudoconvex:

- $\psi \in C^3$, $\nabla d \neq 0$ on $\overline{\Omega}$,

$$\{p_\beta, \{p_\beta, \psi\}\}(x, t, \xi) > 0, \quad \beta \in \{\mu, \lambda + 2\mu\}$$

  for $(x, t, \xi) \in \overline{Q} \times (\mathbf{R}^4 \setminus \{0\})$ satisfying $p_\beta(x, \xi) = (\partial_{\xi_0} p_\beta)(\partial_t \psi) + < \nabla p_\beta, \nabla \psi >= 0$.

- $$\frac{1}{2\sqrt{-1}s} \left\{ p_\beta(x, \xi - \sqrt{-1}s\nabla_{x,t}\psi), p_\beta(x, \xi + \sqrt{-1}s\nabla_{x,t}\psi(x, t)) \right\} > 0, \quad \beta \in \{\mu, \lambda + 2\mu\}$$

  if $s > 0$ and $(x, t, \xi) \in \overline{Q} \times (\mathbf{R}^4 \setminus \{0\})$ satisfying

$$p_\beta(x, \xi + \sqrt{-1}s\nabla_{x,t}\psi)$$

$$= \partial_{\xi_0} p_\beta(x, \xi + \sqrt{-1}s\nabla_{x,t}\psi)\partial_t \psi + < \nabla_\xi p_\beta(x, \xi + \sqrt{-1}s\nabla_{x,t}\psi), \nabla \psi >= 0.$$

Observability inequality:

$$\begin{cases} \rho(x)\partial_t^2 u(x,t) = L_{\lambda,\mu}(x)u(x,t) - \int_0^t L_{\widetilde{\lambda},\widetilde{\mu}}(x,t,\eta)u(x,\eta)d\eta, & \text{in } \Omega \times (0,T), \\ u|_{\partial\Omega} = 0, \\ u(\cdot,0) = 0, \quad \partial_t u(\cdot,0) = a. \end{cases}$$

Theorem 1 (observability inequality).
Let $\Gamma \supset \{x \in \partial\Omega;\ (\nabla d(x) \cdot \nu(x)) \geq 0\}$ and $T > T_0(\rho, \lambda, \mu, \Omega, \psi)$: large. Then

$$\|\partial_t u(\cdot,0)\|_{H^1(\Omega)} \leq C(\|\nabla_{x,t}\partial_\nu u\|_{L^2(0,T;L^2(\Gamma))} + \|\partial_\nu^2 u\|_{L^2(0,T;L^2(\Gamma))})$$

Proof is routine by the Carleman estimate.

$$\begin{cases} \rho(x)\partial_t^2 y(x,t) = L_{\lambda,\mu}(x)y - \int_0^t L_{\widetilde{\lambda},\widetilde{\mu}}(x,t,\eta)y(x,\eta)d\eta + R(x,t)f(x) \\ \qquad \text{in } \Omega \times (0,T), \\ y|_{\partial\Omega} = 0, \quad y(\cdot,0) = \partial_t y(\cdot,0) = 0. \end{cases}$$

$R(x,t)$: 3×3, f: $\mathbf{R}^3$-valued

Inverse source problem. R: given, $\Gamma \supset \{x \in \partial\Omega;\ (\nabla d(x) \cdot \nu(x)) \geq 0\}$

Determine f in Ω by $\partial_\nu y|_{\Gamma\times(0,T)}$.

Theorem 2 (inverse source problem).

Let det $|R(\cdot,0)| \neq 0$ and $\partial_t R(\cdot,0) = 0$ on $\overline{\Omega}$, $T > T_0(\rho,\lambda,\mu,\Omega,\psi)$: large. Then

$$\|f\|_{H^1(\Omega)} \leq C(\|\nabla_{x,t}\partial_\nu y\|_{H^1(0,T;L^2(\Gamma))} + \|\partial_\nu^2 y\|_{H^1(0,T;L^2(\Gamma))})$$

for $f \in H_0^1(\Omega)$.

§4. Key Carleman estimate

$\Box_{\rho,\beta} := \rho(x)\partial_t^2 - \beta(x)\Delta$

$$\begin{cases} \rho(x)\partial_t^2 y - L_{\lambda,\mu} y + \int_0^t L_{\widetilde{\lambda},\widetilde{\mu}} y(x,\eta)d\eta = F, \\ \partial_t^j y(\cdot, \pm T) = 0, \quad j = 0, 1. \end{cases}$$

Let

$$\begin{cases} v_1 := \mathrm{rot}\, y - \int_0^t \frac{\widetilde{\mu}(x,t,\eta)}{\mu(x)} \mathrm{rot}\, y(x,\eta)d\eta, \\ v_2 := \mathrm{div}\, y - \int_0^t \frac{(\widetilde{\lambda}+2\widetilde{\mu})(x,t,\eta)}{(\lambda+2\mu)(x)} \mathrm{div}\, y(x,\eta)d\eta, \end{cases}$$

$\Longrightarrow$

$$\begin{cases} \Box_{\rho,\mu} v_1 = [\text{lower order terms of } \nabla_{x,t} v_1, \nabla_{x,t} v_2] + \int_0^t [\Delta v_1, \Delta v_2] d\eta, \\ \Box_{\rho,\lambda+2\mu} v_2 = [\text{lower order terms of } \nabla_{x,t} v_1, \nabla_{x,t} v_2] + \int_0^t [\Delta v_1, \Delta v_2] d\eta, \\ \text{Boundary condition} : \mathcal{B}(v_1, v_2) = g \end{cases}$$

Theorem 3 (key Carleman estimate).

Let $Q := \Omega \times (-T, T)$, $v := (v_1, v_2) \in H^1(Q)$,

Let $\varphi(x,t) := e^{\tau\psi}$, $\tau > 0$: large, $\psi(x,t) = d(x) - \beta t^2$, $\beta > 0$: small

$$\int_Q (s|\nabla v|^2 + s^3|v|^2)e^{2s\varphi}\,dxdt$$

$$\leq C\int_Q (|\mathrm{rot}\,F|^2 + |\mathrm{div}\,F|^2 + |F|^2)e^{2s\varphi}\,dxdt + Ce^{Cs}(\|\nabla_{x,t}\partial_\nu y\|^2_{L^2(\Gamma\times(-T,T))} + \|\partial_\nu^2 y\|_{L^2}).$$

for all large $s > 0$.

Here $\rho(x)\partial_t^2 y - L_{\lambda,\mu}y + \int_0^t L_{\widetilde{\lambda},\widetilde{\mu}}y(x,\eta)d\eta = F$ and $\partial_t^j y(\cdot, \pm T) = 0$ with $j = 0, 1$.

$$\begin{cases} v_1 := \mathrm{rot}\,y - \int_0^t \dfrac{\widetilde{\mu}(x,t,\eta)}{\mu(x)}\,\mathrm{rot}\,y(x,\eta)d\eta, \\ v_2 := \mathrm{div}\,y - \int_0^t \dfrac{(\widetilde{\lambda}+2\widetilde{\mu})(x,t,\eta)}{(\lambda+2\mu)(x)}\,\mathrm{div}\,y(x,\eta)d\eta, \end{cases}$$

Our Carleman estimate does not assume compact support for y. $\Longrightarrow$ Lipschitz stability:

$\|\text{coefficient}\| \leq C\|\text{Data}\|_{*(\partial\Omega\times(0,T))}$

With Carleman estimate for compact supports, we can get only Hölder stability

Partial references on Hölder stability:

1. de Buhan-Osses (2010)
2. Lorenzi-Messina-Romanov (2007), Lorenzi-Romanov (2006)
3. Romanov-Yamamoto (2010)

Recipe for Proof of Carleman estimate for Lamé system

- Decoupling
- Reduce integral term to lower term

No general theory for Carleman estimate for strongly coupled systems:

$\rho(x)\partial_t^2 y = \mu(x)\Delta y + (\lambda + \mu)\nabla \mathrm{div}\, y +$lower order terms

Decoupling into weakly coupled system (by Fichera, etc.):

References.

[1] Imanuvilov-Yamamoto: ESAIM CO **11** (2005) pp.1-56

[2] Imanuvilov-Yamamoto: Lec. Note Pure Appl. Math. **242** (2005), Chapman, pp.337-374

[3] Imanuvilov-Yamamoto: Publ. Res. Inst. Math. Sci. **43** (2007), pp.1023-1093

[4] Imanuvilov-Yamamoto: Appl. Anal. **88** (2009), pp.711-733

Recipe for integral term

Cavaterra-Lorenzi-Yamamoto: 2006

$$\partial_t^2 u = p(x)\Delta u + \int_0^t K(x,t,\eta)\Delta u(x,\eta)d\eta + F$$

Key invertible integral transform:

$v(x,t) = p(x)u(x,t) + \int_0^t K(x,t,\eta)u(x,\eta)d\eta \Longrightarrow$

$$\partial_t^2 v = p(x)\Delta v + \widetilde{P_1}u(x,t) + \int_0^t K(x,t,\eta)\widetilde{P_2}u(x,\eta)d\eta_{\text{=lower term}} + pF$$

Here $\widetilde{P_j}$: first-order operators

Key Lemma (Klibanov):

$\varphi(x,t)$: maximum at $t = t_0$. Then

$$\int_Q \left| \int_{t_0}^{t} |w(x,\eta)| d\eta \right|^2 e^{2s\alpha} dxdt \leq \frac{C}{s} \int_Q |w|^2 e^{2s\varphi(x,t)} dxdt.$$

This lemma + hyperbolic Carleman estimate $\Longrightarrow$

Theorem 0.

Let $u(\cdot, 0) = 0$ or $\partial_t u(\cdot, 0) = K(\cdot, 0, 0) = 0$. Then

$$\int_Q (s|\nabla_{x,t} u|^2 + s^3|u|^2)e^{2s\alpha} dxdt \leq C \int_Q |F|^2 e^{2s\alpha} dxdt$$

for all large $s > 0$ and supp $u \subset \Omega \times (0, T)$ satisfying

$$\partial_t^2 u = p(x)\Delta u + \int_0^t K(x, t, \eta)\Delta u(x, \eta) d\eta + F$$

§5. Determination of viscoelastic coefficients

$u_k(a)$

$$\begin{cases} \partial_t^2 u_k = L_{\lambda,\mu}(x)u_k + \int_0^t L_{\widetilde{\lambda}_k,\widetilde{\mu}_k}(x,t,\eta)u_k(x,\eta)d\eta, \\ \qquad \text{in } \Omega \times (-T,T), \\ u_k|_{\partial\Omega} : given, \\ u_k(\cdot,0) = a, \quad \partial_t u_k(\cdot,0) = 0, \quad k = 1,2. \end{cases}$$

Here $\widetilde{\lambda}_k(x,t,\eta) = \ell_k(x)p(t,\eta)$, $\widetilde{\mu}_k(x,t,\eta) = m_k(x)q(t,\eta)$.

p, q: known, $p(0,0), q(0,0) \neq 0$.

Coefficient inverse problem with two measurements.

$$\|\ell_1 - \ell_2\| + \|m_1 - m_2\| \leq C \sum_{i=1}^{2} (\|\partial_\nu(u_1(a_i) - u_2(a_i))\|?$$

Coefficient inverse problem $\Longrightarrow y := u_1 - u_2 \Longrightarrow$ Inverse source problem

Non-degeneracy condition similar to det $R(\cdot, 0) \neq 0$ for inverse source problem requires conditions for a_1, a_2.

Theorem 4.

Assume $\ell_k, m_k \in C^4(\overline{\Omega})$, $\ell_k, m_k, \partial_\nu \ell_k, \partial_\nu m_k$: given on $\partial\Omega$,

$\|u_k\|_{W^{8,\infty}(\Omega\times(-T,T))} \leq M$: a priori bound

Let

- $T > T_0$ (large constant).
- Let a pair of initial displacement a_1, a_2 satisfy non-degeneracy condition (*).

$\Longrightarrow$

$$\|\ell_1 - \ell_2\|_{H^1(\Omega)} + \|m_1 - m_2\|_{H^1(\Omega)}$$

$$\leq C \sum_{i=1}^{2} (\|\nabla_{x,t}\partial_\nu(u_1(a_i) - u_2(a_i))\|_{H^2(0,T;L^2(\Gamma))} + \|\partial_\nu^2(u_1(a_i) - u_2(a_i))\|_{H^2(L^2)})$$

Non-degeneracy condition (*) for initial data a_1, a_2

For 6×4 matrix A, we set
$\{A\}_{i,j}$: matrix created by deleting i-th and j-th rows of A,
$\det_{i,j} A = \det\{A\}_{i,j}$
E_3: 3×3 identity matrix.

Non-degeneracy condition (*) $\Longleftrightarrow$ $\exists i_1, i_2, j_1, j_2 \in \{1, 2, \ldots, 6\}$ such that

$$\det_{i_1,j_2} \begin{pmatrix} (\mathrm{div}\, a_1)E_3 & (\nabla a_1 + (\nabla a_1)^T)\nabla d \\ (\mathrm{div}\, a_2)E_3 & (\nabla a_2 + (\nabla a_2)^T)\nabla d \end{pmatrix} \neq 0,$$

$$\det_{j_1,j_2} \begin{pmatrix} \nabla a_1 + (\nabla a_1)^T & (\mathrm{div}\, a_1)\nabla d \\ \nabla a_2 + (\nabla a_2)^T & (\mathrm{div}\, a_2)\nabla d \end{pmatrix} \neq 0$$

on $\overline{\Omega}$.

Example of a_1, a_2. Let $\Omega \subset \{(x_1, x_2, x_3);\ x_1, x_3 > 0\}$, $d(x) = |x|^2$,

$$p(t,\eta) = q(t,\eta) =: \widetilde{p}(t-\eta), \quad \widetilde{p}(0) = 1.$$

Then $a_1 = \begin{pmatrix} 0 \\ x_1 x_2 \\ 0 \end{pmatrix}$ and $a_2 = \begin{pmatrix} x_1 \\ 0 \\ x_3 \end{pmatrix}$ satisfy (*):

(*) $\Longrightarrow$

$$\det_{2,4} \begin{Bmatrix} x_1 & 0 & 0 & 2x_2^2 \\ 0 & x_1 & 0 & 6x_1x_2 \\ 0 & 0 & x_1 & 0 \\ 2 & 0 & 0 & 4x_1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 4x_3 \end{Bmatrix} = -8x_1^2 x_3 \neq 0 \quad \text{on } \overline{\Omega}.$$

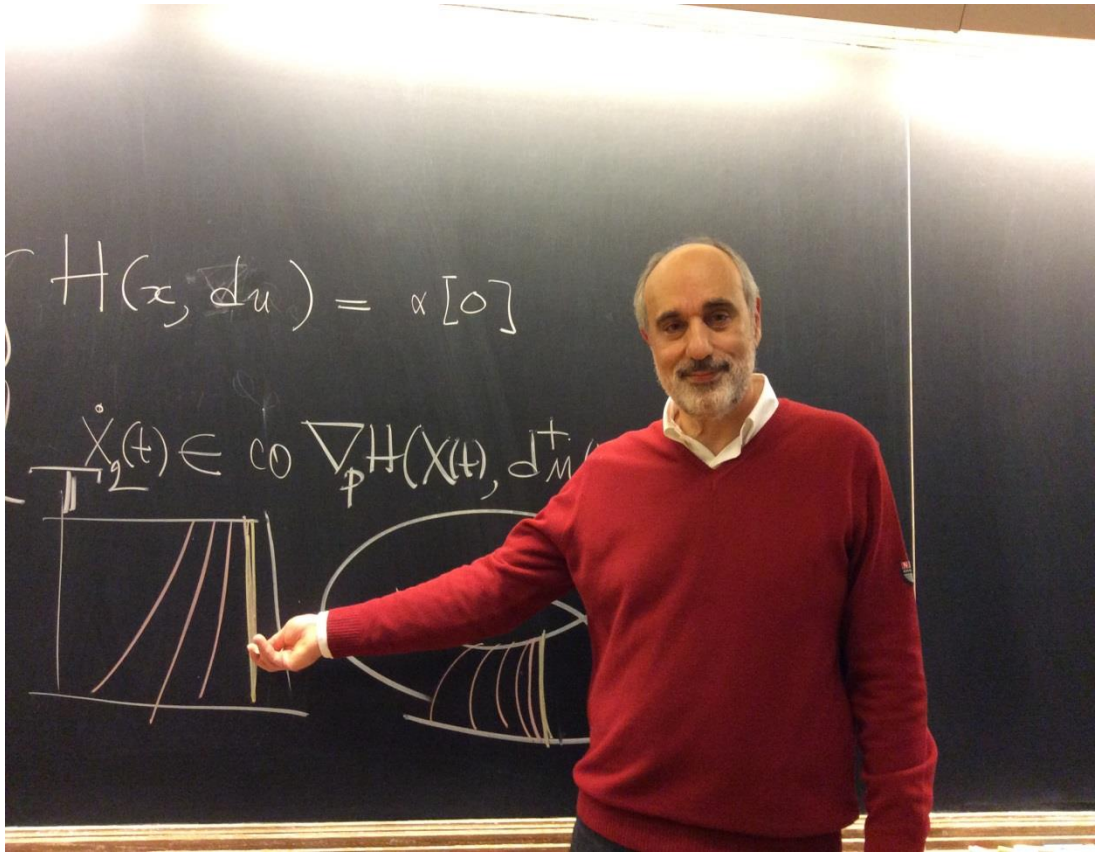
$$\det_{3,5} \begin{Bmatrix} 0 & x_2 & 0 & 2x_1^2 \\ x_2 & 2x_1 & 0 & 2x_1x_2 \\ 0 & 0 & 0 & 2x_1x_3 \\ 2 & 0 & 0 & 4x_1 \\ 0 & 0 & 0 & 4x_2 \\ 0 & 0 & 2 & 4x_3 \end{Bmatrix} = 16x_1^3 \neq 0 \quad \text{on } \overline{\Omega}.$$

Main messages

With data on sub-boundary,

Carleman estimate for viscoelasticity equations yields

- Partial observability inequality
- Lipschitz stability for inverse problems of spatial varying Lamé coefficients



Age 60 = one cycle of the life calendar

Life Begins At Sixty !

PIERMARCO,
CONGRATULATIONS
ON THE 60TH BIRTHDAY
GOOD HEALTH!