$d(PMC, EZ) = 2$

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CANNARSA 60, Rome, July 2017

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So many reasons...

$$d(PMC, EZ) = 2$$

Proof 1:

$$d(PMC, EZ) = d(PMC, JV) + d(JV, EZ) = 1 + 1 = 2$$

Proof 2:

$$d(PMC, EZ) = d(PMC, VK) + d(VK, EZ) = 1 + 1 = 2$$

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ESAIM:COCV, 1999: Cannarsa, Komornik & Loreti

ESAIM: Control, Optimisation and Calculus of Variations April 1999, Vol. 4, p. 37–56
URL: http://www.emath.fr/cocv/

WELL POSEDNESS AND CONTROL OF SEMILINEAR WAVE EQUATIONS WITH ITERATED LOGARITHMS *

Piermarco Cannarsa[1], Vilmos Komornik[2] and Paola Loreti[3]

Abstract. Motivated by a classical work of Erdős we give rather precise necessary and sufficient growth conditions on the nonlinearity in a semilinear wave equation in order to have global existence for all initial data. Then we improve some former exact controllability theorems of Imanuvilov and Zuazua.

Résumé. Motivé par un travail classique d'Erdős on donne des conditions nécessaires et suffisantes de croissance de la non linéarité dans une équation des ondes semilinéaire pour l'existence des solutions globales pour toutes les données initiales. Ensuite on améliore certains théorèmes antérieurs de contrôlabilité exacte de Imanuvilov et de Zuazua.

Roughly they establish the controllabililty of the $1-d$ semilinear wave equation for nonlinearities of the form

$$f(s) = s[\log[\log[\log \ldots |s|]]]^2$$

Proof [2]

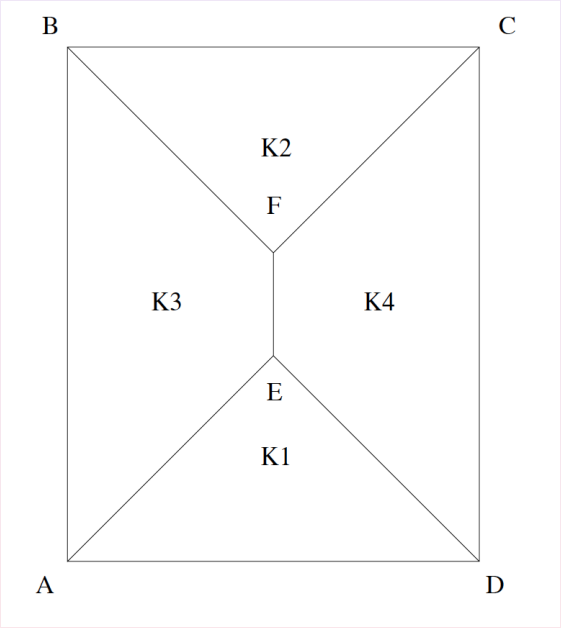


Figure 2 in their paper

[2] Similar sidewise energy arguments have been used in other contexts such as waves on networks: J. Lagnese, G. Leugering and G. Schmidt...

Why?

The nonlinearity

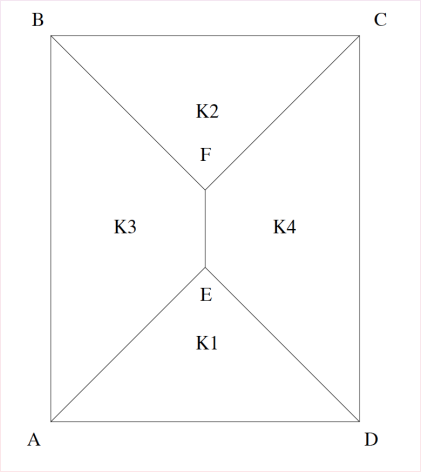
$$f(s) = s[\log[\log[\log \ldots |s|]]]^2$$

is *subcritical* both for

$$y_{tt} - y_{xx} + f(y) = 0$$

and

$$y_{xx} - y_{tt} - f(y) = 0$$



Osgood condition

For first order ODEs

$$x'(t) = f(x(t))$$

the criticality condition is

$$\int_{z_0}^{\infty} \frac{1}{f(z)} dz = \infty.$$

This condition ensures global well-posedness, while finite-time blow up occurs if the integral converges.

Thus

$$f(s) = s[\log[\log[\log \ldots |s|]]]$$

is **critical**.

2nd order Osgood condition

For second order ODEs

$$x''(t) = f(x(t))$$

multiplying by $x'(t)$ we get

$$\frac{1}{2}|x'(t)|^2 - F(x(t)) = C$$

And therefore

$$x'(t) = \pm\sqrt{2(C + F(x(t))},$$

the critical threshold becomes

$$f(s) = s[\log[\log[\log \ldots |s|]]]^2.$$

Multi-d

The multi-d analog of this result is unknown, because of the ill-posedness of the sidewise wave equation.
A partial result is known for non-linearities of the order of [3]

$$|f(s)| << |s| \log^{1/2}(|s|)|, \text{ as } |s| \to \infty.$$

Employing techniques based on Carleman inequalities one could expect to extend this result to the range

$$|f(s)| << |s| \log^{3/2}(|s|)|, \text{ as } |s| \to \infty,$$

as it occurs for the heat equation.

But getting to the 1-d range seems hard although nothing excludes the result to be true, as far as we know.

[3] X. Zhang and E. Z. Exact controllability of the semi-linear wave equation. In "Unsolved problems in mathematical systems and control theory", Princeton University Press, 2004, pp. 173-178.

This 2/3 exponent may not be improved through Carleman estimates as pointed out by Th. Duyckaerts, X. Zhang and EZ (Annales IHP, 2005), based on the following result by V. Z. Meshkov, 1991.

Theorem

(Meshkov, 1991). *Assume that the space dimension is $n = 2$. Then, there exists a nonzero complex-valued bounded potential $q = q(x)$ and a non-trivial complex valued solution $u = u(x)$ of*

$$\Delta u = q(x)u, \qquad in\ \mathbb{R}^2, \tag{1}$$

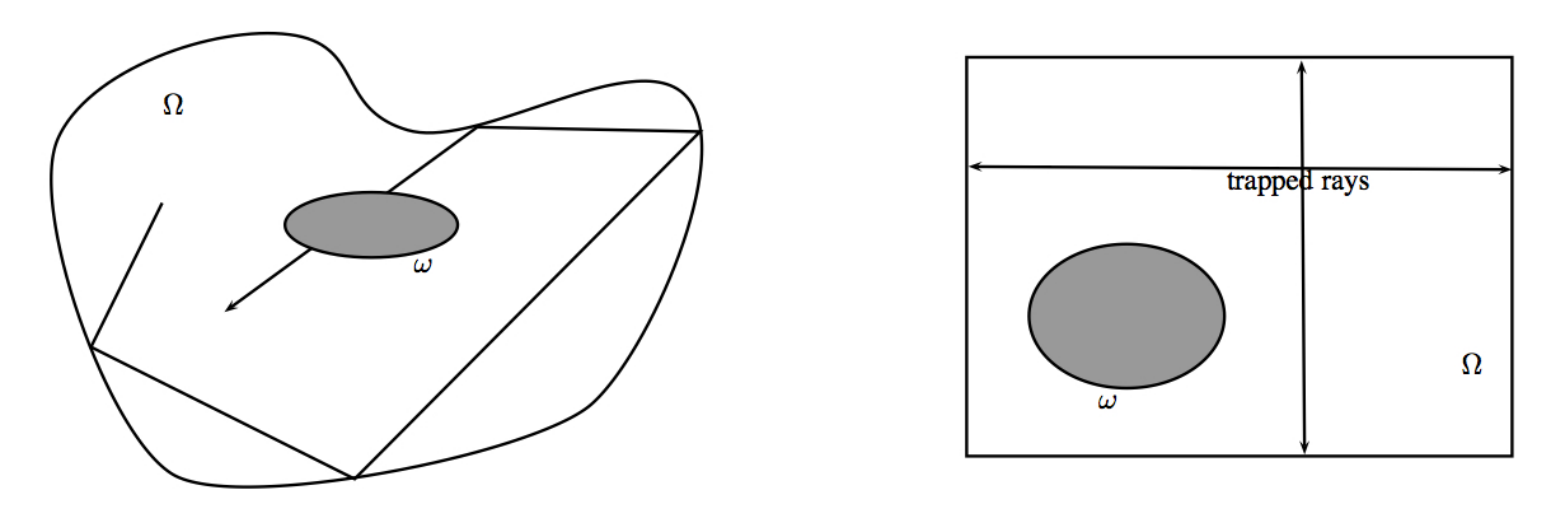
with the property that

$$\mid u(x) \mid \leq C \exp(- \mid x \mid^{4/3}), \qquad \forall\, x \in \mathbb{R}^2 \tag{2}$$

for some positive constant $C > 0$.

What if the control time is too short?

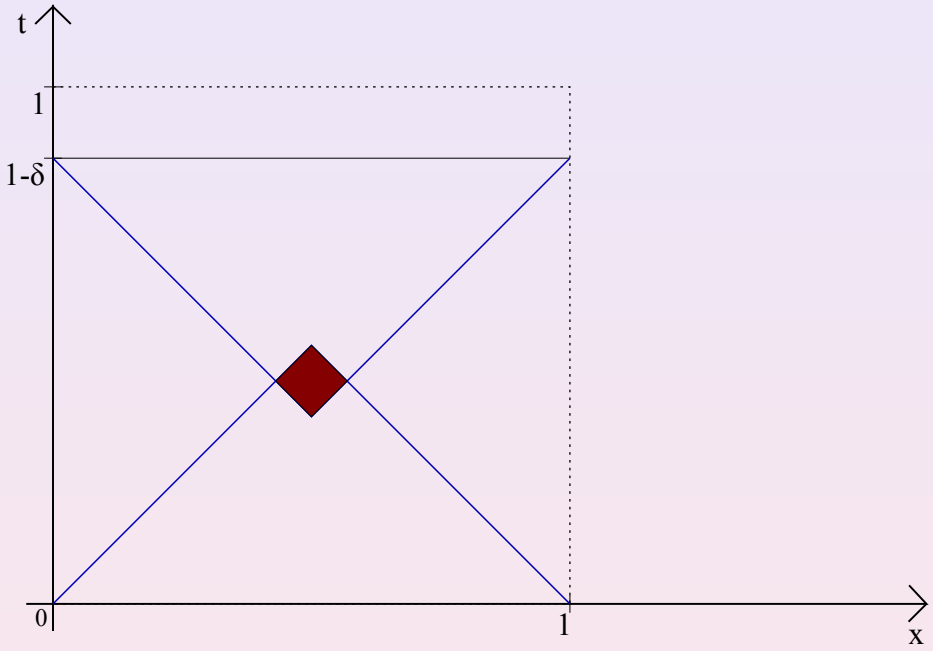
These results hold for large enough time horizons of control. Roughly, under the so-called Geometric Control Condition (GCC) by Bardos - Lebeau - Rauch (1988). It asserts, roughly, that all rays of geometric optics enter the control set ω in time T.



When this condition holds all data can be controlled.
But what when T is too short for GCC to be fulfilled?

1-d : $T < 2$

In 1-d if $T < 2$ the space of controllable initial data can be characterised either using characteristic arguments or Fourier series expasions:[4] [5]



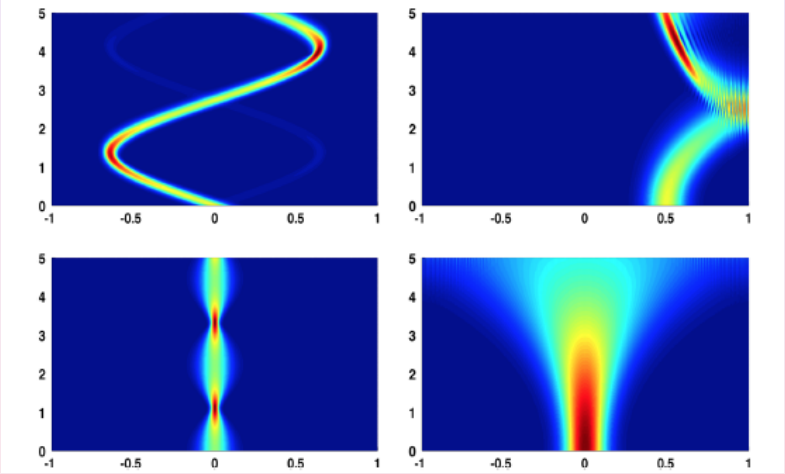
[4] J. Lohéac and E. Zuazua, Norm saturating property of time optimal controls for wave-type equations, CPDE 2016, Bertinoro, 2016, IFAC-PapersOnLine, 49 (8) 37-42, 2016.

[5] D. A. Ivanov, M. M. Potapov. Approximations to Time-Optimal Boundary Controls for Weak Generalized Solutions of the Wave Equation. Computational Mathematics and Mathematical Physics, 2017, Vol. 57, No. 4, pp. 607-625.

Numerical algorithms

How to build numerical algorithms enabling to compute the minimal control time for a specific initial datum?

Note that, so far, all effort have been devoted to build numerical algorithms enabling to get the GCC. But nothing has been done for shorter T's.[6], [7]



[6] E. Z. Propagation, observation, and control of waves approximated by finite difference methods. SIAM Review, 47 (2) (2005), 197-243.

[7] A. Marica y E. Zuazua, Propagation of $1-D$ waves in regular discrete heterogeneous media: A Wigner measure approach, Found. Comput. Math. (2015) 15:1571-1636.

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Motivation

Beyond standard control problems in which the control enters as an external applied source, many problems arising in population dynamics, ecology, etc raise interesting questions about the possibility of controlling the dynamics regulating some of the parameters entering in the system. [8], [9]

Approximate controllability of

$$y_t - y_{xx} + v(x,t)y + f(y) = 0,$$

employing intermitent alternating controls: From initial to final data with the same number of zeroes...

1. Phase 1: $v \equiv 0$, free evolution of the semilinear equation.
2. Phase 2: $v \equiv v(x)$, static control.

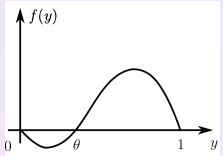
[8] P. M. Cannarsa, G. Floridia, A.Y. Khapalov, Multiplicative controllability for semilinear reaction-diffusion equations with finitely many changes of sign, arxiv 2015, JMPA, to appear.

[9] Motivated by tumor growth, fissionable materials,...

Numerical control of travelling wave solutions: Jiamin Zhu (LJLL-Paris6) and E. Z.

The initial value problem

$$y_t - y_{xx} = ay(1-y)(y-\theta), \quad y(0,\cdot) = y_0, \quad x \in \mathbf{R}, \quad t \in \mathbf{R}^+,$$

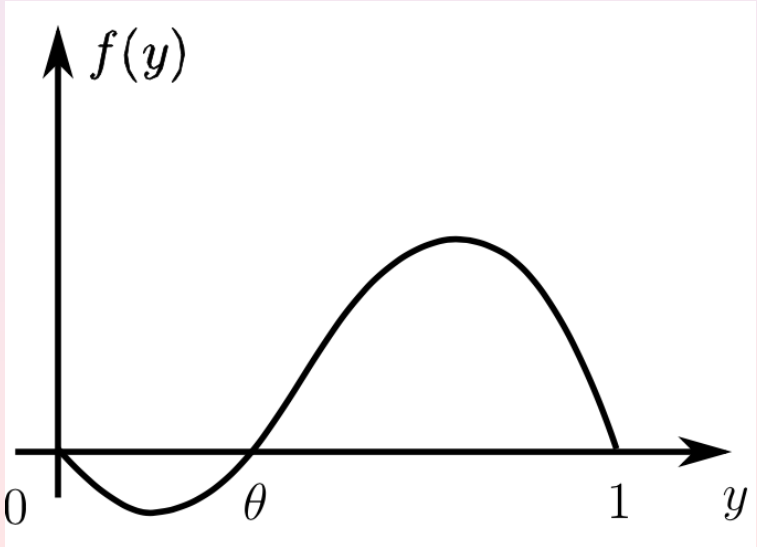


- Represents the net population change from birth and death.
- Typical application : spread of invading organisms in ecology systems, (cf. e.g. M.A. Lewis and P. Kareiva 1993).
- The role of parameteres a, θ:
  - $a > 0$: reproductive rate
  - $\theta \in (0,1)$: local critical density or Allee threshold that determines the sign (positive or negative) the population growth.
- Other applications : population genetics (biology), propagation of nerve tension (neurobiology), propagation waves in chemical reactors (chemistry), etc.

Our original motivation: multilinguism

A Game-Theoretic Analysis of Minority Language Use in Multilingual Societies[1]

José-Ramón Uriarte
University of the Basque Country
October 2015

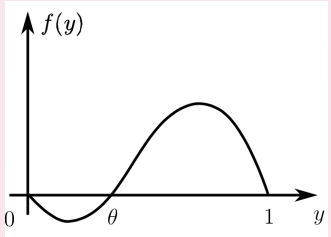


Allee effet : population growth is negative (leading to the extinction) when the density of the population is lower than the Allee threshold θ, otherwise the population will reach carrying capacity.

- increase θ by the sterile male technique, the mating disruption (pest management technique), etc.
- decrease θ by providing protection (e.g. efficient feeding, suppressing natural enemies) to the population

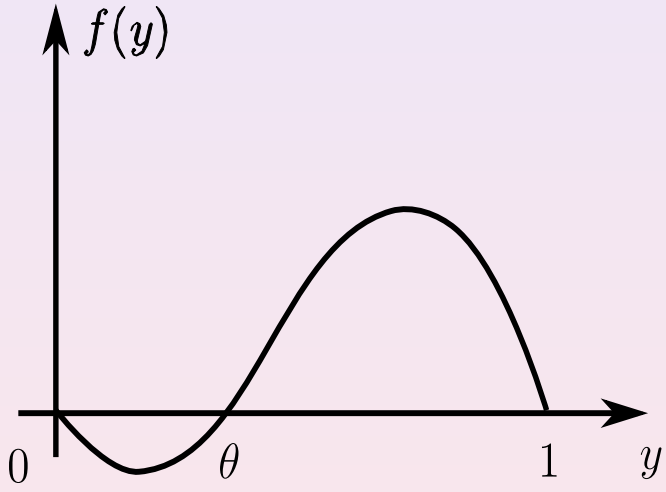
Typical solutions of the system :

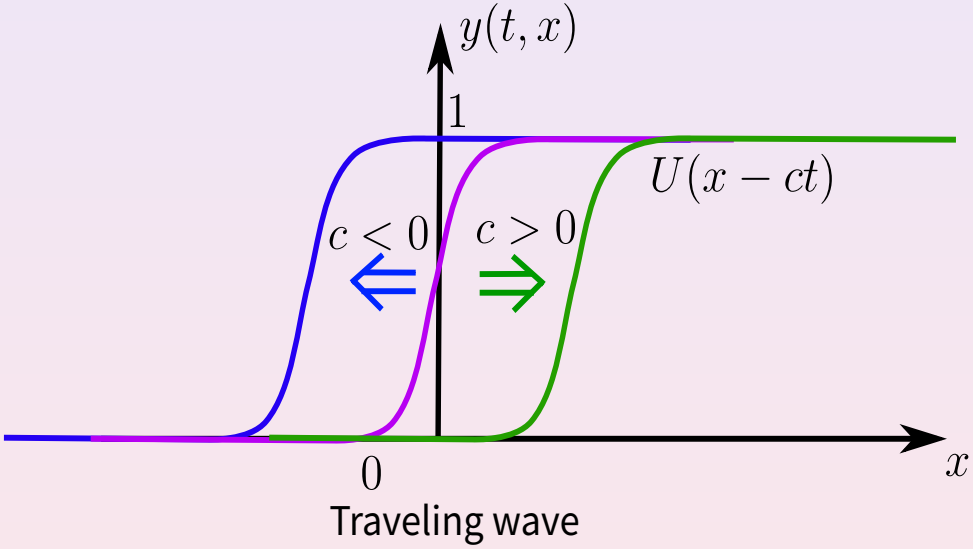
- Steady state constant solutions : $y \equiv 0$ or θ or 1.
- Traveling wave solutions : link two of the three constant solutions



Traveling wave solutions

- solution of the form $y(t,x) = U(x-ct)$, $U(\pm\infty) = U_\pm$, $U(\pm\infty)' = 0$,
  where $U(x)$ is the wave profile and c is the wave speed.
- sign of the wave speed : $\mathrm{sign} c = -\int_0^1 f(t)dt$



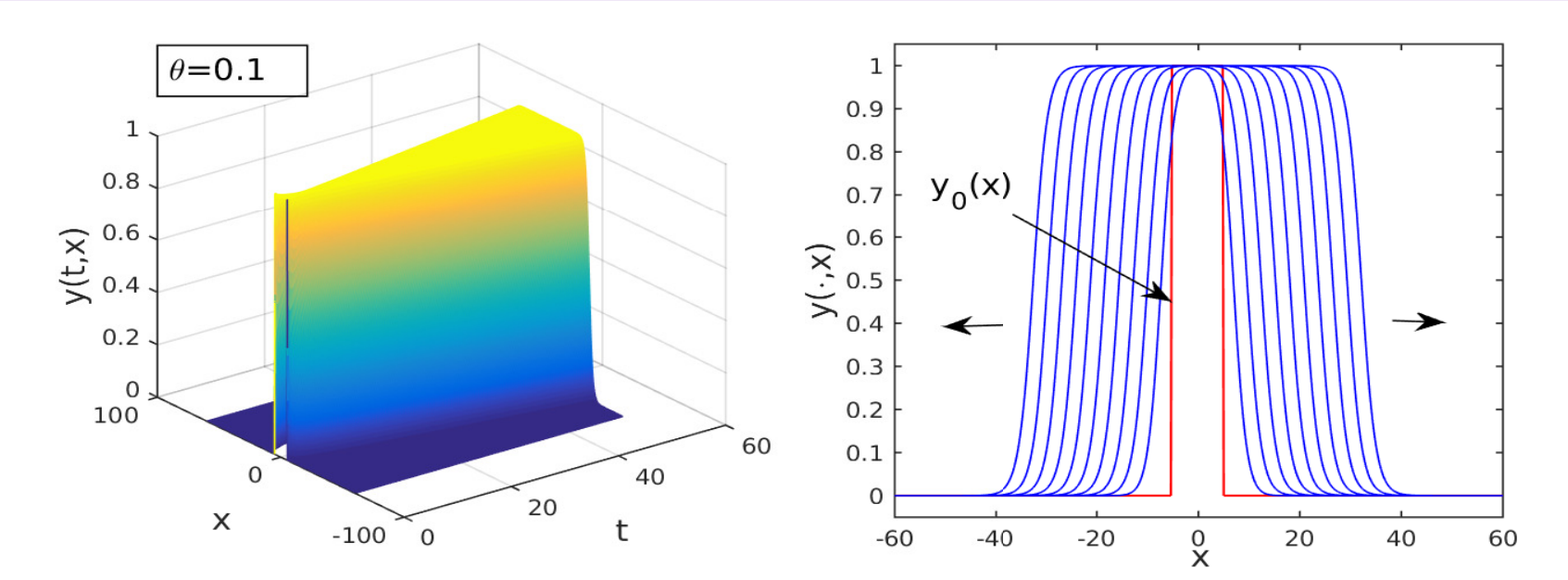


- Effect of control θ
  - change the wave speed
  - when $U_- = 0$, $U_+ = 1$, $c = \sqrt{a/2}(2\theta - 1)$ and $U(x) = \frac{e^{\sqrt{a/2}x}}{1+e^{\sqrt{a/2}x}}$

La ola

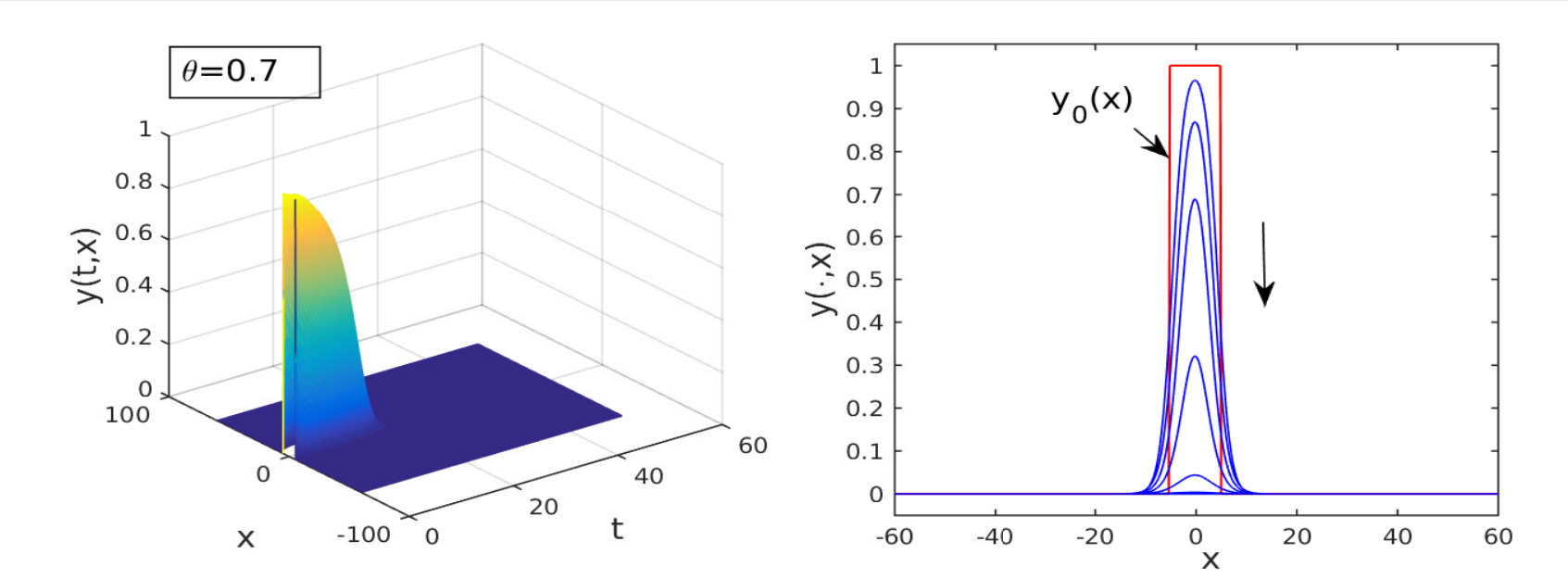
Is it possible to control the traveling waves ?

- $\theta = 0.1$: invasion of the population



Is it possible to control the traveling waves ?

- $\theta = 0.1$: invasion of the population
- $\theta = 0.7$: extinction of the population



Formulation of the control problem

Control problem $\mathcal{P}_c$

Find $\theta(t) \in [0,1]$, $t \in [0, T]$ such that the solution of

$$y_t - y_{xx} = ay(1-y)(y-\theta),$$

$$y(0,\cdot) = y_0(\cdot)$$

develops into an expected wave $U(\cdot)$ at the given time T, minimizing

$$J(\theta) = \left| y(T,\cdot) - U \right|^2$$

.

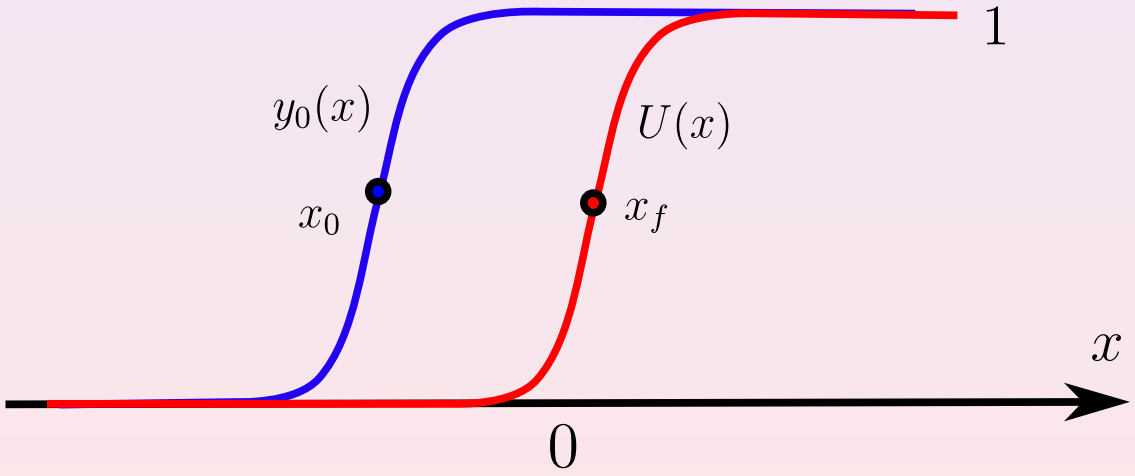
Explicit solutions : from traveling wave to traveling wave

- $U(x - c_d t)$ is a traveling wave : given $\theta_d \in (0,1)$, $x_d \in \mathbf{R}$

$$U(x) = e^{\sqrt{a/2}(x-x_f)}/(1 + e^{\sqrt{a/2}(x-x_f)}),$$

$$c_d = \sqrt{a/2}(2\theta_d - 1)$$

- $y_0(\cdot)$ is a translation of $U(\cdot)$, i.e., $y_0 = U(x + x_f - x_0)$



Wave speed c is bounded $\Rightarrow$ T must be sufficiently large, i.e., $T \geq T_{min}$

Piecewise constant control strategy (infinite many choices)

$$\theta^*(t) = \begin{cases} \theta^1, & t \in (0, t_1), \\ \theta^2, & t \in [t_1, t_1 + h), \\ \cdots & \\ \theta^N, & t \in [T - h, T], \end{cases}$$

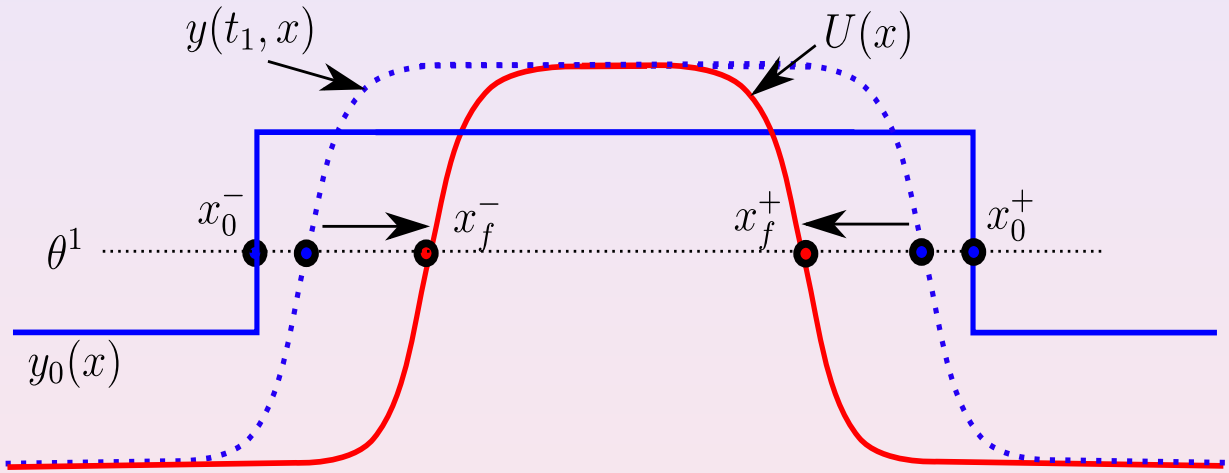
Choose θ^i, $i = 2, \cdots, N$ such that $h \sum_{i=2}^{N} c(\theta^i) = (x_f - x_0' - c_1 t_1)$, where $c(\theta^i) = \sqrt{a/2}(2\theta^i - 1)$

Since $c(\theta^i) < \sqrt{|a_1/2|} \Rightarrow T$ should be sufficiently large, i.e.,

$$T > T_{min} = t_1 + \frac{|x_f - x_0' - c_1 t_1|}{\sqrt{|a_1/2|}}$$

Quasi explicit examples

One can use the existing results on the stability and the attractiveness of the dynamics for $\theta = \textit{constant}$ fixed to "approximately" control the system in two steps.[10],[11]



[10] D.G. Aronson, H.F. Weinberger, Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation, in PDE and Related Topics, Lecture Notes in Math., vol. 446, Springer, Berlin, 1975, pp. 5-49.

[11] P.C. Fife, J.B. McLeod, The approach of solutions of nonlinear diffusion equations to travelling front solutions, ARMA, 1977, vol. 65 no. 4, p. 335-361.

Numerics: Gradient descent

- Start with an initialization $u^0 = (\theta^1, \cdots, \theta^N, t_1)$.
- At step k :
  - step 1 : integrate the state system with control u^k from $t = 0$, $y(0, \cdot) = y_0$ to $t = T \Rightarrow$ output $y(t, x)$
  - step 2 : compute the cost $J(u^k)$. If $J(u^k) < \epsilon$, stop.
  - step 3 : integrate the adjoint system from $t = T$, $p(T, \cdot) = 2(\bar{y}(T, \cdot) - Y_d)$ to $t = 0 \Rightarrow$ output $p(t, x)$
  - step 4 : calculate the gradient of the cost functional by $J'(u^k) = \int_0^T \int_\Omega p(t, x) \partial_u f(y, u^k) dx dt$, where p is the adjoint state.
  - step 5 : set the descend direction $d_k = J(u^k)$, compute a descend step size s_k, and set the new control by $u^{k+1} = u^k + s_k d_k$. Return to step 1.
- non convex optimal control problem $\Rightarrow$ numerically obtained optimal controls depend on the initialization u^0

Two-grids

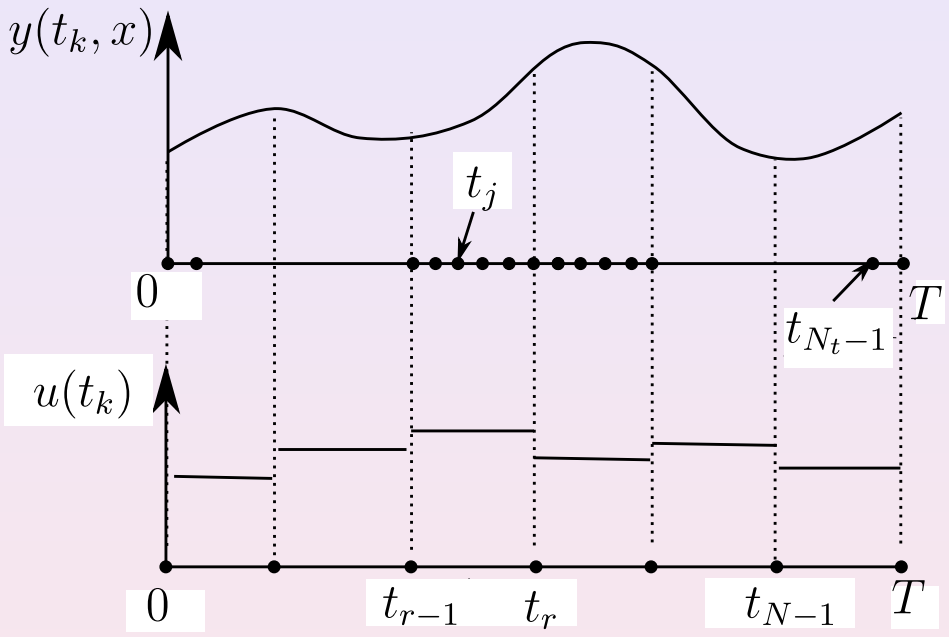


Figure: Space subdivisions for the state y and for the control u at time grid t_k.

Numerical experiments: Fighting against extintion, $T > T_{min}$

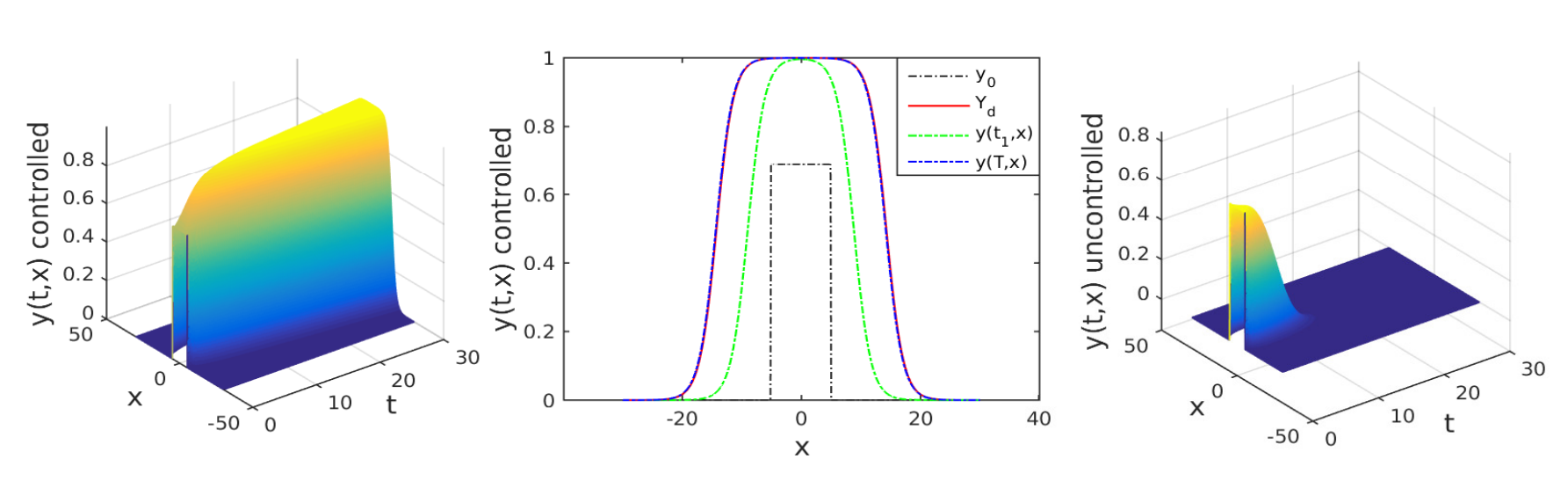


Figure: Left and middle : solution of problem $(\mathcal{P})_{opt}$ after control;
Right : state y of uncontrolled system.

Numerical experiments: Fighting against extintion, $T < T_{min}$

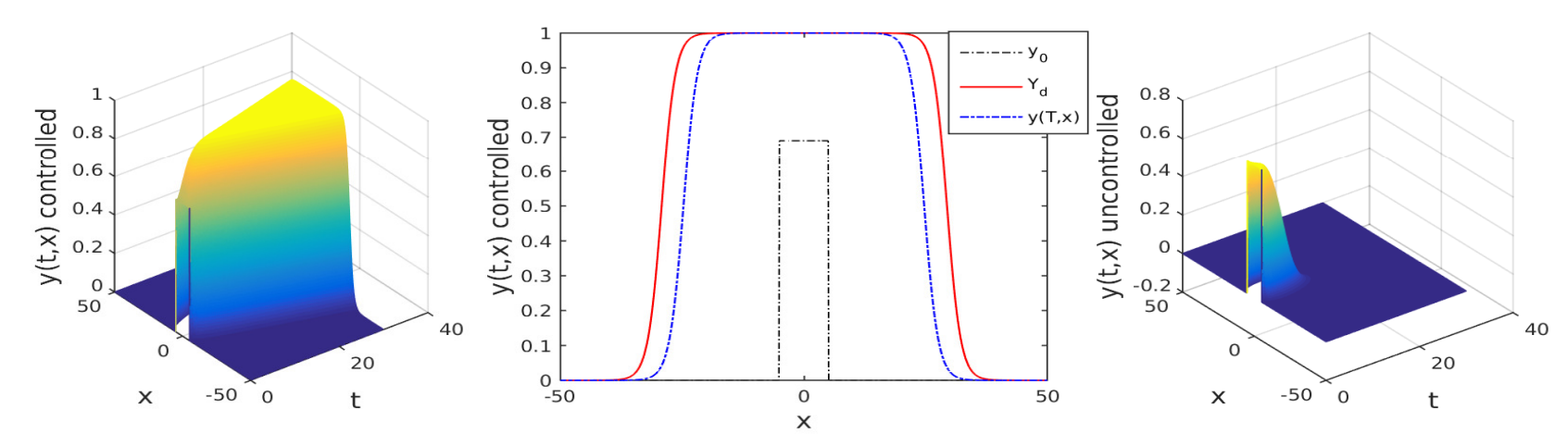


Figure: Left and middle : solution of problem $(\mathcal{P})_{opt}$ after control;
Right : state y of uncontrolled system.

Perspectives

- So far numerical simulations: Theoretical analysis to be done beyond quasi-explicit examples.
- More complex models (multi-d, systems,...)

Happy birthday, Piermarco. We will continue the celebration in Benasque August 20th...