A new comparison between overconvergent isocrystals and arithmetic $\mathcal{D}^!$-modules
joint with Tomoyuki Abe

Christopher Lazda
Warwick Mathematics Institute

Padova, 19th September 2019
1 Introduction

2 Constructible isocrystals

3 The trace map

4 Dual constructibility
\( \mathcal{V} = \) complete DVR
\( k = \) residue field, perfect, char \( p > 0 \)
\( K = \) fraction field \( K \), char 0

\( X/k \) variety ( = separated scheme of finite type), \( \text{Isoc}^\dagger(X/K) = \) \( F \)-able
overconvergent isocrystals on \( X \)

If we have an embedding \( X \hookrightarrow \mathfrak{P} \) with \( \mathfrak{P} \) smooth and proper over \( \mathcal{V} \), then

\[
\text{Isoc}^\dagger(X/K) \hookrightarrow \text{MIC}(j_\mathfrak{X}_!\mathcal{O}_{Y[1]}),
\]

where \( Y \) is the closure of \( X \) inside \( \mathfrak{P}_k \), and

\[
H^i_{\text{rig}}(X/K, E) := H^i(Y[1], E \otimes \Omega^\bullet_{Y[1]})
\]

Good formal properties: finite dimensional, versions with support, excision exact
sequences, &c.
Beyond “smooth” coefficient objects: theory of arithmetic $\mathcal{D}^\dagger$-modules (Berthelot/Caro).

Locally: take étale co-ordinates $x_1, \ldots, x_d$ on $\mathcal{V}$ and set

$$\mathcal{D}_{\mathcal{V}Q}^\dagger = \left\{ \sum_k a_k \partial[k] \left| a_k \in \mathcal{O}_{\mathcal{V}Q}, \exists \lambda > 1 \text{ s.t. } \|a_k\| \lambda^{|k|} \to 0 \right. \right\}$$

where

$$\partial[k] = \frac{\partial_{x_1}^{k_1} \cdots \partial_{x_d}^{k_d}}{k_1! \cdots k_d!}.$$

Caro defines

$$D^b_{\text{hol}}(\mathcal{D}_{\mathcal{V}Q}^\dagger) \subset D^b_{\text{coh}}(\mathcal{D}_{\mathcal{V}Q}^\dagger)$$

“$F$-able overholonomic complexes”, stable under:

- $f^!$
- $D_{\mathcal{V}}$
- $\otimes_{\mathcal{V}Q}^\dagger$
- $R\Gamma^\dagger_Z$ for $Z \subset \mathcal{V}$ closed
- $f_+$ for $f$ proper,
Given $X \hookrightarrow \mathcal{P}$, define

$$D_{\text{hol}}^b(X/K) := \{ \mathcal{M} \in D_{\text{hol}}^b(\mathcal{D}_{\mathbb{P}^1_{\mathbb{Q}}}) \text{ supported on } X \} ,$$

these are independent of the embedding, and support a formalism of the 6 functors $(f^+, f_+), (f_!, f^!), \otimes, \mathbb{D}$.

Comparison of coefficients: $\exists$ fully faithful functor

$$\text{sp}_{X,+} : \text{Isoc}^\dagger(X/K) \to D_{\text{hol}}^b(X/K) \subset D_{\text{hol}}^b(\mathcal{D}_{\mathbb{P}^1_{\mathbb{Q}}})$$

and can describe the essential image. Defined by Caro when $X$ is smooth, and extended to the non-smooth case by Abe.

**Example**

If $X$ is a dense open inside $P = \mathbb{P}_k$, and $P \setminus X$ is a divisor, then $\text{sp}_+ = \text{sp}_*$ is just pushforward along $\text{sp} : \mathbb{P}_k \to \mathbb{P}$.

Much more difficult to describe in general!
For $E \in \text{Isoc}^\dagger(X/K)$, can define its “$\mathcal{D}$-module cohomology”

$$H^i_{\mathcal{D}}(X/K, E) := H^{i-d_X}(f_+sp_+E)$$

where $f : X \to \text{Spec}(k)$ is the structure morphism, inducing

$$f_+ : D^b_{\text{hol}}(X/K) \to D^b_{\text{hol}}(\text{Spec}(k)/K) \cong D^b(K),$$

and $d_X = \dim X$. Concretely, if $X \xhookrightarrow{\mathfrak{p}}$ then

$$H^i_{\mathcal{D}}(X/K, E) = H^{i-d_X+d_{\mathfrak{p}}}(\mathfrak{p}, sp_+E \otimes_{\mathcal{O}_{\mathfrak{p}}} \Omega^\bullet_{\mathfrak{p}/\nu}).$$

where $d_{\mathfrak{p}} = \dim \mathfrak{p}$.

**Question**

Do we always have

$$H^i_{\text{rig}}(X/K, E) \cong H^i_{\mathcal{D}}(X/K, E)?$$

This is not obvious!

Today: describe a new construction of $sp_+$ which makes comparison theorems easier to prove.
1 Introduction

2 Constructible isocrystals

3 The trace map

4 Dual constructibility
\( \mathfrak{P} = \) smooth, proper formal \( \mathcal{V} \)-scheme, \( P = \) special fibre,

\[
p_1, p_2 : ]P[_{\mathfrak{P}^2} \to \mathfrak{P}_K
\]

the two projections.

**Definition (Berthelot)**

A convergent stratification on an \( \mathcal{O}_{\mathfrak{P}_K} \)-module \( E \) is an isomorphism

\[
p_2^* E \sim p_1^* E
\]

satisfying the cocycle condition.

**Definition (Le Stum)**

\( E \) is called constructible if there exists a stratification \( P = \bigsqcup_i P_i \) such that \( E|_{P_i} \) is coherent.

\( \text{Isoc}^\dagger_{\text{cons}}(P/K) = (F\text{-able}) \) constructible \( \mathcal{O}_{\mathfrak{P}_K} \)-modules with convergent stratification.
Example

$X \hookrightarrow \mathcal{V}$ locally closed immersion, with closure $\alpha : Y \hookrightarrow \mathcal{V}$, then we have $\lfloor \alpha \rfloor ! Y \to \mathcal{V}_K$, and if $E \in \text{Isoc}^\dagger(X/K) \subset \text{MIC}(j_X^! \mathcal{O}_Y)$, then

$\lfloor \alpha \rfloor ! E \in \text{Isoc}^\dagger_{\text{cons}}(P/K)$

so we have a fully faithful functor

$\lfloor \alpha \rfloor ! : \text{Isoc}^\dagger(X/K) \to \text{Isoc}^\dagger_{\text{cons}}(P/K)$.

Conjecture (Le Stum)

$Rsp_\ast$ induces an equivalence of categories

$\text{Isoc}^\dagger_{\text{cons}}(P/K) \sim \text{Perv}(\mathcal{D}^\dagger_{\mathcal{V}/\mathbb{Q}})$.

This is a theorem when $\dim \mathcal{V}/\mathcal{V} = 1$. 
Not clear how to even define $R_{sp_*}$ in general: want a lifting

$$\xymatrix{ D^b(\mathcal{D}_Q^\dagger) \ar[r] & D^b(\mathcal{D}_Q^\dagger) \\
\text{Isoc}^\dagger_{\text{cons}}(P/K) \ar[r]_{R_{sp_*}} \ar[d]^\text{forget} & D^b(\mathcal{D}_Q^\dagger) \\
\text{Isoc}^\dagger_{\text{cons}}(P/K) \ar[r]_{R_{sp_*}} \ar[u] & D^b(\mathcal{D}_Q^\dagger) \ar[u] }$$

but $sp^{-1}\mathcal{D}_Q^\dagger$ doesn’t act on constructible isocrystals (even if they are coherent on all of $\mathfrak{B}_K$). What we did: given $X \hookrightarrow Y \overset{\alpha}{\twoheadrightarrow} \mathfrak{B}$, construct

$$\xymatrix{ \text{Isoc}^\dagger(X/K) \ar[r]_-{[\alpha]} & \text{Isoc}^\dagger_{\text{cons}}(P/K) \ar[r]_{R_{sp_*}} \ar[d]^\text{forget} & D^b(\mathcal{D}_Q^\dagger) \\
\text{Isoc}^\dagger(X/K) \ar[r]_-{[\alpha]} & \text{Isoc}^\dagger_{\text{cons}}(P/K) \ar[r]_{R_{sp_*}} \ar[u] & D^b(\mathcal{D}_Q^\dagger) \ar[u] }$$

This immediately gives

$$H^i_{\text{rig}}(P, [\alpha][!E]) \cong H^i_{\mathcal{D}}(\mathfrak{B}, R_{sp_*}[\alpha][!E]).$$
Basic example: \( \mathcal{P} = \widehat{\mathbb{P}}^2 \) with co-ordinates \( x_0, x_1, x_2 \), \( Y = \mathbb{P}^1_k = \mathbb{V}(x_2) \subset P \), and 
\( X = \mathbb{A}^1_k = D(x_0) \subset Y \), so we have

\[
X \xrightarrow{j} Y \xrightarrow{\alpha} \mathcal{P}
\]

and

\[
\alpha[\cdot]Y[\to \mathcal{P}_K.
\]

We take \( E = j^!_{\mathcal{X}} \mathcal{O}_Y[\mathcal{P}] \).

Set \( U = P \setminus Y \), so for any sheaf \( \mathcal{F} \) on \( \mathcal{P}_K \) we have the localisation exact sequence

\[
0 \to \Gamma_Y^! \mathcal{F} \to \mathcal{F} \to j^!_U \mathcal{F} \to 0,
\]

note that \( \Gamma_Y^! = \alpha[\cdot] \alpha^{-1} \). We apply this to \( \mathcal{F} = R\alpha[*j^!_{\mathcal{X}} \mathcal{O}_Y[ \mathcal{P}] \) to obtain

\[
0 \to \alpha[i^!j^!_{\mathcal{X}} \mathcal{O}_Y[ \to \alpha[*j^!_{\mathcal{X}} \mathcal{O}_Y[ \to j^!_U \alpha[*j^!_{\mathcal{X}} \mathcal{O}_Y[ \to 0
\]

which gives a 2-term resolution of \( \alpha[i^!j^!_{\mathcal{X}} \mathcal{O}_Y[. \)
Lemma

The sheaves $\alpha[\ast j_X^! O_X]_Y$ and $j_U^! \alpha[\ast j_X^! O_X]_Y$ are $\text{sp}_*\text{-acyclic}$.

So we have

$$R\text{sp}_* \alpha[j_X^! O_X]_Y \cong (\text{sp}_* \alpha[j_X^! O_X]_Y \to \text{sp}_* j_U^! \alpha[j_X^! O_X]_Y).$$

If we set $u = x_1/x_0$ and $v = x_2/x_0$, and look at global sections, then the first term consists of series

$$f(u, v) \in K[[u, v]]$$

such that:

- $\forall \eta < 1 \exists \lambda > 1$ s.t. $f(u, v)$ converges for $|v| \leq \eta$ and $|u| \leq \lambda$.

Can describe the second term similarly, as series

$$f(u, v) \in K[[u, v, v^{-1}]]$$

such that:

- there exists $\rho < 1$ such that $\forall \rho < \eta < 1 \exists \lambda > 1$ s.t. $f(u, v)$ converges for $\rho \leq |v| \leq \eta$ and $|u| \leq \lambda$. 

Christopher Lazda
Explicitly, the second is Kedlaya’s relative Robba ring $\mathcal{R}^\nu_{K\langle u \rangle^\dagger}$, and the first is its plus part $\mathcal{R}^{\nu,+}_{K\langle u \rangle^\dagger}$ consisting of series with terms in non-negative powers of $\nu$.

$\Rightarrow$ can see directly that

$$\mathbf{Rsp}_* \alpha[i j^\dagger_X \mathcal{O}_Y \mathbb{Y}] \cong \nu^{-1} K\langle u, \nu^{-1} \rangle^\dagger[-1]$$

and the $\mathcal{D}_\mathbb{P}_\mathbb{Q}$-module structure extends to a $\mathcal{D}_\mathbb{P}_\mathbb{Q}$-module structure.

Want to generalise this to an arbitrary frame $(X \xrightarrow{j} Y \xrightarrow{\alpha} \mathbb{P})$ with $\mathbb{P}$ smooth and proper over $\mathcal{Y}$, and $E \in \text{Isoc}^\dagger(X/K)$. 
Three complications:

1. The complement \( Y \setminus X \) might not be a divisor. So we take a suitable open cover \( X = \bigcup_a X_a \) and replace \( E \) by

\[
0 \to \oplus a j^\dagger_{X_a} E \to \oplus a_1, a_2 j^\dagger_{X_{a_1} \cap X_{a_2}} E \to \ldots \to j^\dagger_{\bigcap_a X_a} E \to 0.
\]

2. We don’t know in general that the \( j^\dagger_{X_a} E \) are \( \alpha [\ast] \)-acyclic. So we take the immersions \( \alpha_\eta : [Y]_\eta \to [Y] \) of quasi-compact tubes, and replace \( j^\dagger_{X_a} E \) by

\[
\lim_{\rightarrow n_0} \left( \prod_{n \geq n_0} \alpha_{\eta_n \ast j^\dagger_{X_a} E│[Y]_{\eta_n}} \to \prod_{n \geq n_0} \alpha_{\eta_n \ast j^\dagger_{X_a} E│[Y]_{\eta_n}} \right)
\]

for \( \eta_n \to 1^- \).

3. \( Y \) might not be a divisor in \( P \), so we need to pick divisors \( D_b \) such that \( Y = \bigcap_b D_b \) and replace the short exact sequence

\[
0 \to \Gamma^\dagger_Y \mathcal{F} \to \mathcal{F} \to j^\dagger_{U} \mathcal{F} \to 0
\]

by the long exact sequence

\[
0 \to \Gamma^\dagger_Y \mathcal{F} \to \mathcal{F} \to \oplus b j^\dagger_{U \setminus D_b} \mathcal{F} \to \ldots \to j^\dagger_{U \setminus \bigcup_b D_b} \mathcal{F} \to 0.
\]
Proposition

Given suitable choices $X = \bigcup_a X_a$ and $Y = \bigcap_b D_b$ as above there exists a resolution $RC^\dagger(E)$ of $]\alpha[! E$ such that the $\mathcal{D}_{\mathcal{P}\mathcal{Q}}$-module structure on $sp_*RC^\dagger(E)$ extends canonically to a $\mathcal{D}_{\mathcal{P}\mathcal{Q}}^\dagger$-module structure.

Changing the $X_a$ or the $D_b$ results in canonically quasi-isomorphic complexes of $\mathcal{D}_{\mathcal{P}\mathcal{Q}}^\dagger$-modules.

Corollary

There exists a canonical lifting of $(\text{R}sp_* \circ ]\alpha[!) [d_{\mathcal{P}}]$ to a functor

$$\text{R}sp_{\mathcal{P}^\dagger, !} : \text{Isoc}^\dagger (X/K) \to D^b(\mathcal{D}_{\mathcal{P}\mathcal{Q}}^\dagger)$$

such that

$$H_{\text{rig}}^i (P, ]\alpha[! E) = H^{i-2d_{\mathcal{P}}} (u_+ \text{R}sp_{\mathcal{P}, !} E).$$

Example

If $Y = P$ and $Y \setminus X$ is a divisor, then $\text{R}sp_{\mathcal{P}, !} E = sp_* E[d_{\mathcal{P}}] = sp_+ E[d_{\mathcal{P}}].$
Introduction

Constructible isocrystals

The trace map

Dual constructibility
Suppose we want to interpret

\[ H^i_{\text{rig}}(P, \alpha[!E]) = H^i(\mathfrak{M}_K, \alpha[!((E \otimes \Omega^\bullet_{\mathcal{Y}[]})]. \]

Morally, the RHS should be

\[ "H^i_{\text{c}}(\mathcal{Y}[, E \otimes \Omega^\bullet_{\mathcal{Y}[]})" \]

for some suitable definition of \( H^i_{\text{c}} \).

So we want to:

1. make sense of \( H^i_{\text{c}} \) for rigid analytic varieties (following Huber);
2. use a suitable formalism of the trace map to understand \( H^i_{\text{c}}(\mathcal{Y}[, E \otimes \Omega^\bullet_{\mathcal{Y}[]}) \) via \textit{duality} (following the approaches of Chiarellotto/van der Put to Serre duality for rigid analytic varieties).
Proposition

For every paracompact (taut) morphism $f : V \to W$ of rigid analytic varieties there exists a unique functor

$$R_f! : D^b(Ab(V)) \to D^b(Ab(W))$$

such that:

- when $f$ is partially proper, $R_f!$ is the total derived functor of the functor $f_!$ of sections whose support is quasi-compact over $W$ (in particular, when $f$ is proper, $R_f! = Rf_*$);
- when $f$ is an open immersion, $R_f! = f_!$ is extension by zero;
- $R(g \circ f)_! = Rg_! \circ Rf_!$.

Moreover, for any $w \in W$ there is a canonical isomorphism

$$(Rf_! -)_w \xrightarrow{\sim} R\Gamma_c(f^{-1}(w), -).$$

Remark

When $f$ is partially proper, we recover the definition given by van der Put.
By considering more general adic spaces, we can define $Rf_!$ by these properties: by a theorem of Huber every such $f : V \to W$ has a canonical compactification

![Diagram](attachment:image.png)

where $j$ is an open immersion and $\bar{f}$ is partially proper. Then define $Rf_! := R\bar{f}_! \circ j_!$ where $R\bar{f}_!$ is the total derived functor of sections with quasi-compact support.
Key new input:

**Theorem (Proper base change theorem)**

Let $f : V \to W$ be a proper morphism of finite dimensional adic spaces, and $\mathcal{F} \in \text{Ab}(V)$. Then for every $w \in W$ the base change map

$$(Rf_* \mathcal{F})_w \to R\Gamma(f^{-1}(w), \mathcal{F})$$

is an isomorphism.

**Proof.**

Reduce to $W = \text{Spa}(L, L^+)$ for some affinoid field $(L, L^+)$, $V = \mathbb{P}^1_{(L,L^+)}$, $w =$ closed point of $W$, and $\mathcal{F} =$ constant sheaf $A_T$ supported on some closed subset $T \subset \mathbb{P}^1_{(L,L^+)$. Then use an explicit topological description of $\mathbb{P}^1_{(L,L^+)}. \hfill \square$
Theorem

There exists a unique way to associate a trace morphism

\[ \text{Tr}_f : Rf_!\Omega^\bullet_{V/W}[2d] \to \mathcal{O}_W \]

to every smooth, paracompact morphism \( f : V \to W \) of rigid analytic varieties, of relative dimension \( d \), such that:

1. \( \text{Tr}_f \) is compatible with composition;
2. when \( f \) is étale, then \( \text{Tr}_f \) is the canonical map
   \[ f_!\mathcal{O}_V \to \mathcal{O}_W; \]
3. when \( W = \text{Sp}(R) \) is affinoid, and \( f : \mathbb{D}_W(0; 1^-) \to W \) is the canonical projection, then \( \text{Tr}_f \) is induced by the map
   \[ H^1_c(\mathbb{D}_W(0; 1^-), \Omega^1_{\mathbb{D}_W(0; 1^-)/W}) \cong R\langle z^{-1}\rangle^\dagger d \log z \to R \]
   \[ \sum_{i \geq 0} r_i z^{-i} d \log z \mapsto r_0 \]

where \( z \) is any co-ordinate on \( \mathbb{D}_W(0; 1^-) \).

If \( f \) is either a \( \mathbb{D}^n(0; 1^-) \) or \( \mathbb{A}^{n,\text{an}} \)-bundle, then \( \text{Tr}_f \) is an isomorphism.
To construct $\text{Tr}_f$:

- Properties (1) and (3) give a trace map for $f : \mathbb{D}_W^n(0; 1^-) \rightarrow W$, at least given a choice of co-ordinates $z_1, \ldots, z_d$.

- Properties (1) and (2) then give a trace map for $f : \mathbb{D}_W^n(0; 1) \rightarrow W$, at least given a choice of $z_1, \ldots, z_d$ and a choice of uniformiser $\pi \in \mathcal{V}$ inducing

$$\mathbb{D}_W^n(0; 1) \xrightarrow{j_\pi} \mathbb{D}_W^n(0; 1^-).$$

- Properties (1) and (2) then give a trace map whenever $W$ and $V$ are affinoids, at least given a choice of factorisation

$$V \xrightarrow{g} \mathbb{D}_W^n(0; 1) \rightarrow W$$

with $g$ étale.

- Can construct $\text{Tr}_f$ in general by using descent.

The hard work is in proving independence of all of these choices!
Corollary

Let

\[
\begin{array}{ccc}
Y' & \rightarrow & Y' \\
\downarrow & & \downarrow \\
g & \rightarrow & u \\
\downarrow & & \downarrow \\
X & \rightarrow & Y & \rightarrow & Y \\
\end{array}
\]

be a diagram of frames, with \( g \) proper and \( u \) smooth in a neighbourhood of \( X \), and \( E \in \text{Isoc}^+(X/K) \). Then \( \text{Tr}_g[ \] induces an isomorphism

\[
\mathbb{R}g_! [E]_{Y'[\mathfrak{P}']} \otimes \Omega^\bullet_{Y'[\mathfrak{P}']/\mathfrak{P}} [2d_u] \sim E_{\mathfrak{Y}'[\mathfrak{P}]}
\]

where \( d_u \) is the relative dimension.

Corollary

Let \((X, Y \overset{\alpha}{\rightarrow} \mathfrak{P})\) be a smooth and proper frame over \( \mathcal{V} \), and \( E \in \text{Isoc}^+(X/K) \). Then

\[
H^{2d_{\mathfrak{P}}-i}(\mathfrak{P}_K, \alpha[1] E \otimes \Omega^\bullet_{\mathfrak{P}_K})
\]

only depends on \( X \) and not on \( Y \) or \( \mathfrak{P} \).
Now, if we take a smooth and proper frame \((X, Y, \mathfrak{P})\) and \(E \in \text{Isoc}^\dagger(X/K)\), then we have Berthelot’s “Poincaré” pairing

\[
E \times R\Gamma_{\mathcal{X}[\mathfrak{p}]} E^\vee \to R\Gamma_{\mathcal{X}[\mathfrak{p}]} \mathcal{O}_Y[\mathfrak{p}]
\]

which via the trace map induces a pairing

\[
H^{2d\mathfrak{p} - i}_{c}(\mathcal{Y}[\mathfrak{p}], E \otimes \Omega^\bullet_{\mathcal{Y}[\mathfrak{p}]}) \times H^{i}(\mathcal{Y}[\mathfrak{p}], R\Gamma_{\mathcal{X}[\mathfrak{p}]} (E^\vee \otimes \Omega^\bullet_{\mathcal{Y}[\mathfrak{p}]}) )
\]

\[
\to H^{2d\mathfrak{p}}_{c}(\mathcal{Y}[\mathfrak{p}], \Omega^\bullet_{\mathcal{Y}[\mathfrak{p}]}) \stackrel{\text{Tr}}{\to} K.
\]

**Theorem**

*This pairing is perfect.*

**Proof.**

Both sides sit in excision exact sequences, which are compatible with the pairing, so we may assume that \(X\) is smooth and affine. We can therefore choose a Monsky–Washnitzer frame \((X, Y, \mathfrak{P})\) in which case the claim reduces to Poincaré duality with coefficients, as proved by Kedlaya.

So \(H_{\text{rig}}^{2d-\mathfrak{p} - i}(P, \alpha[!E])\) is canonically isomorphic to rigid Borel–Moore homology \(H_{i,\text{rig}}^{BM}(X, E)\).
1 Introduction

2 Constructible isocrystals

3 The trace map

4 Dual constructibility
Constructible isocrystals

The trace map

Dual constructibility

\[ V = \text{complex variety} \sim 2 \text{ well-known } t\text{-structures of } D^b_c(V, \mathbb{C}). \]

1. The usual (constructible) \( t\)-structure \((D^{\geq 0}, D^{\leq 0})\) with heart \( \text{Con}(V, \mathbb{C})\).
2. The perverse \( t\)-structure \((^pD^{\geq 0}, ^pD^{\leq 0})\) with heart \( \text{Perv}(V, \mathbb{C})\).

Second is self-dual under \( D_V \), first is not.

Definition

The dual constructible \( t\)-structure \((^dD^{\geq 0}, ^dD^{\leq 0})\) on \( D^b_c(V, \mathbb{C})\) is defined by

\[
\mathcal{K}^\bullet \in ^dD^{\geq 0} \iff D_V(\mathcal{K}^\bullet) \in D^{\leq 0} \\
\mathcal{K}^\bullet \in ^dD^{\leq 0} \iff D_V(\mathcal{K}^\bullet) \in D^{\geq 0}.
\]

Deduce properties of \((^dD^{\geq 0}, ^dD^{\leq 0})\) from those of \((D^{\geq 0}, D^{\leq 0})\).

Example

If \( f : V \to W \) then \( f^! \) is exact for the dual constructible \( t\)-structure. If \( f \) is an immersion, then so is \( Rf_* \).
Back to char $p$: $\sim D^b_{\text{hol}}(\mathcal{D}^\dagger_{\mathcal{P}Q})$, $D^b_{\text{hol}}(X/K)$ also have 3 $t$-structures:

1. holonomic $t$-structure - on $D^b_{\text{hol}}(\mathcal{D}^\dagger_{\mathcal{P}Q})$ this is just the obvious one coming from $D^b_{\text{coh}}(\mathcal{D}^\dagger_{\mathcal{P}Q})$, slightly more subtle on $D^b_{\text{hol}}(X/K)$;
2. constructible (perverse) $t$-structure;
3. dual constructible $t$-structure.

Same exactness properties as before, in particular the dual constructible $t$-structure on $D^b_{\text{hol}}(X/K)$ is the restriction of that on $D^b_{\text{hol}}(\mathcal{D}^\dagger_{\mathcal{P}Q})$ - this is false for the other two!

**Remark**

When $\mathcal{P}$ is a smooth and proper curve, Le Stum's perverse $t$-structure on $D^b_{\text{hol}}(\mathcal{D}^\dagger_{\mathcal{P}Q})$ coincides with our dual constructible $t$-structure, up to a shift by $1 = \dim \mathcal{P}$.

Hearts are denoted

$$\text{Hol}(\mathcal{P}), \text{Con}(\mathcal{P}), \text{DCon}(\mathcal{P})$$

and

$$\text{Hol}(X/K), \text{Con}(X/K), \text{DCon}(X/K)$$

respectively.
Theorem

Let $X \hookrightarrow \mathcal{Y}$ with $\mathcal{Y}$ smooth and proper over $\mathcal{V}$, and $E \in \text{Isoc}^\dagger(X/K)$. Then

$$R\text{sp}_{\mathcal{Y},!}E \in D\text{Con}(X/K) \subset D^b(\mathcal{D}^\dagger_{\mathcal{Y} \mathcal{Q}})$$

is overholonomic, supported on $X$, and is in the heart of the dual constructible $t$-structure.

Proof.

1. Show that formation of $R\text{sp}_{\mathcal{Y},!}E$ is compatible with localisation exact sequences and taking finite étale covers of $X$ (this uses a suitable $\mathcal{D}^\dagger$-lifting of the trace morphism).

2. Use alterations to reduce to the case where $X$ and $Y := \overline{X}$ are smooth and $Y \setminus X$ is a divisor.

3. Now locally lift $Y \hookrightarrow \mathcal{Y}$ to a closed embedding $u : \mathcal{Z} \hookrightarrow \mathcal{Y}$ of smooth formal $\mathcal{V}$-schemes, and show that $u_+ R\text{sp}_{\mathcal{Z},!}E \cong R\text{sp}_{\mathcal{Y},!}E$, thus reducing to the case when $Y = \mathcal{Y}_k$.

4. In this case we have $R\text{sp}_{\mathcal{Y},!} = \text{sp}_+[d_{\mathcal{Y}}]$ and can appeal to Caro–Tsuzuki.
Proposition

Consider

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
\Psi & \xrightarrow{u} & \Omega
\end{array}
\]

with $\Psi$, $\Omega$ proper smooth over $\mathcal{V}$, and $E \in \text{Isoc}^\dagger(Y/K)$. Then there is a canonical isomorphism

\[
\mathbf{R}\Gamma_X u^! \mathbf{Rsp}_\Omega E \sim \mathbf{Rsp}_\Psi f^* E
\]

in $\mathbf{D}^b_{\text{hol}}(\mathcal{O}_\mathcal{P}/\mathcal{Q})$.

Proof.

We can treat separately the cases when $u = \text{id}$ and the square is Cartesian. The first follows from compatibility with localisation already mentioned, and the second from direct calculation.
Corollary

For any variety $X/k$ there exists a canonical functor

$$\text{sp}_{X,!} : \text{Isoc}^\dagger(X/K) \to \text{DCon}(X/K)$$

such that for any embedding $X \hookrightarrow \mathcal{P}$,

$$(\text{sp}_{X,!}E)_{\mathcal{P}} = \mathbb{R}\text{sp}_{\mathcal{P},!}E \in D^b_{\text{hol}}(\mathcal{D}_{\mathcal{P},\mathbb{Q}}).$$

It is compatible with pullback: for any $f : X \to Y$, and any $E \in \text{Isoc}^\dagger(Y/K)$, we have

$$\text{sp}_{X,!}f^*E \cong f^!\text{sp}_{Y,!}E.$$
Theorem

For any $X/k$, and any $E \in \text{Isoc}^\dagger(X/K)$ there exists a canonical isomorphism

$$\text{sp}_{X,+} E[-d_X] \sim \mathcal{D}_X(\text{sp}_{X,+} E^\vee)$$

in $D^b_{\text{hol}}(\mathcal{D}^\dagger_{\mathbb{Q} \bar{\mathbb{Q}}})$.

Proof.

We can show that both sides lie in the abelian category $\text{Con}(X/K)$, which satisfies h-descent. Hence, we may assume that $X$ is smooth, with a smooth compactification $Y$, and that $Y \setminus X$ is a divisor. Then the isomorphism follows from compatibility of Caro’s functor $\text{sp}_{X,+}$ with duality.

Remark

We only have $\text{sp}_{X,+} E \in \text{Hol}(X/K)$ if $X$ is smooth, in general we have $\text{sp}_{X,+} E \in \text{Con}(X/K)[d_X]$. The formulation of the theorem is slightly neater if we replace $\text{sp}_{X,+} E$ by

$$\tilde{\text{sp}}_{X,+} E := \text{sp}_{X,+} E[-d_X] \in \text{Con}(X).$$
Corollary

For any variety $X/k$, and any $E \in \text{Isoc}^\dagger(X/K)$ we have a canonical isomorphism

$$H^i_{c, \text{rig}}(X/K, E) \cong H^i_{c, \varnothing}(X/K, E) := H^i(f_! \tilde{sp}_X, + E)$$

of $K$-vector spaces.

Proof.

If $X \hookrightarrow Y$ with $Y$ smooth and proper over $V$, and $\alpha : Y \hookrightarrow Y$ is its closure, then we have

$$H^i_{c, \text{rig}}(X/K, E) \cong H^{2d_Y-i}(Y, \alpha[1]E^\vee \otimes \Omega^{\bullet}_{X/K})^\vee$$

$$\cong H^{d_Y-i}(Y, R\text{sp}_Y, !E^\vee \otimes \Omega^{\bullet}_{Y})^\vee$$

$$\cong \text{H}^{-i}(f_+ \tilde{sp}_X, !E^\vee)^\vee$$

$$\cong H^i(f_! \tilde{sp}_X, + E).$$

The general case can be handled by descent.

For comparison of ‘usual’ cohomologies, see Tomoyuki’s talk.
Thank-you!