# A new comparison between overconvergent isocrystals and arithmetic $\mathscr{D}^\dagger$-modules
joint with Tomoyuki Abe

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$\mathcal{V}$ = complete DVR
$k$ = residue field, perfect, char $p > 0$
$K$ = fraction field $K$, char 0

$X/k$ variety (= separated scheme of finite type), $\mathrm{Isoc}^\dagger(X/K)$ = $F$-able overconvergent isocrystals on $X$

If we have an embedding $X \hookrightarrow \mathfrak{P}$ with $\mathfrak{P}$ smooth and proper over $\mathcal{V}$, then

$$\mathrm{Isoc}^\dagger(X/K) \hookrightarrow \mathrm{MIC}(j_X^\dagger \mathcal{O}_{]Y[}),$$

where $Y$ is the closure of $X$ inside $\mathfrak{P}_k$, and

$$H^i_{\mathrm{rig}}(X/K, E) := H^i(]Y[, E \otimes \Omega^\bullet_{]Y[})$$

Good formal properties: finite dimensional, versions with support, excision exact sequences, &c.

Beyond "smooth" coefficient objects: theory of arithmetic $\mathscr{D}^\dagger$-modules (Berthelot/Caro).

Locally: take étale co-ordinates $x_1, \ldots, x_d$ on $\mathfrak{P}$ and set

$$\mathscr{D}^\dagger_{\mathfrak{P}\mathbb{Q}} = \left\{ \sum_{\underline{k}} a_{\underline{k}} \partial^{[\underline{k}]} \;\middle|\; a_{\underline{k}} \in \mathcal{O}_{\mathfrak{P}\mathbb{Q}},\ \exists \lambda > 1 \text{ s.t. } \left\| a_{\underline{k}} \right\| \lambda^{|\underline{k}|} \to 0 \right\}$$

where

$$\partial^{[\underline{k}]} = \frac{\partial_{x_1}^{k_1} \ldots \partial_{x_d}^{k_d}}{k_1! \ldots k_d!}.$$

Caro defines

$$D^b_{\mathrm{hol}}(\mathscr{D}^\dagger_{\mathfrak{P}\mathbb{Q}}) \subset D^b_{\mathrm{coh}}(\mathscr{D}^\dagger_{\mathfrak{P}\mathbb{Q}})$$

"$F$-able overholonomic complexes", stable under:

- $f^!$
- $\otimes^\dagger_{\mathcal{O}_{\mathfrak{P}}}$
- $\mathbf{D}_{\mathfrak{P}}$
- $\mathbf{R}\underline{\Gamma}^\dagger_Z$ for $Z \subset \mathfrak{P}$ closed
- $f_+$ for $f$ proper,

by work of Caro/Caro–Tsuzuki.

Given $X \hookrightarrow \mathfrak{P}$, define

$$D^b_{\mathrm{hol}}(X/K) := \left\{ \mathcal{M} \in D^b_{\mathrm{hol}}(\mathscr{D}^\dagger_{\mathfrak{P}\mathbb{Q}}) \text{ supported on } X \right\},$$

these are independent of the embedding, and support a formalism of the 6 functors $(f^+, f_+), (f_!, f^!), \otimes, \mathbf{D}$.

Comparison of coefficients: $\exists$ fully faithful functor

$$\mathrm{sp}_{X,+} : \mathrm{Isoc}^\dagger(X/K) \to D^b_{\mathrm{hol}}(X/K) \subset D^b_{\mathrm{hol}}(\mathscr{D}^\dagger_{\mathfrak{P}\mathbb{Q}})$$

and can describe the essential image. Defined by Caro when $X$ is smooth, and extended to the non-smooth case by Abe.

**Example**

If $X$ is a dense open inside $P = \mathfrak{P}_k$, and $P \setminus X$ is a divisor, then $\mathrm{sp}_+ = \mathrm{sp}_*$ is just pushforward along $\mathrm{sp} : \mathfrak{P}_K \to \mathfrak{P}$.

Much more difficult to describe in general!

For $E \in \mathrm{Isoc}^\dagger(X/K)$, can define its "$\mathscr{D}$-module cohomology"

$$H^i_{\mathscr{D}}(X/K, E) := H^{i-d_X}(f_+ \mathrm{sp}_+ E)$$

where $f : X \to \mathrm{Spec}\,(k)$ is the structure morphism, inducing

$$f_+ : D^b_{\mathrm{hol}}(X/K) \to D^b_{\mathrm{hol}}(\mathrm{Spec}\,(k)\,/K) \cong D^b(K),$$

and $d_X = \dim X$. Concretely, if $X \hookrightarrow \mathfrak{P}$ then

$$H^i_{\mathscr{D}}(X/K, E) = H^{i-d_X+d_{\mathfrak{P}}}(\mathfrak{P}, \mathrm{sp}_+ E \otimes_{\mathcal{O}_{\mathfrak{P}}} \Omega^\bullet_{\mathfrak{P}/\mathcal{V}}).$$

where $d_{\mathfrak{P}} = \dim \mathfrak{P}$.

**Question**

Do we always have

$$H^i_{\mathrm{rig}}(X/K, E) \cong H^i_{\mathscr{D}}(X/K, E)?$$

This is not obvious!

Today: describe a new construction of $\mathrm{sp}_+$ which makes comparison theorems easier to prove.

1. Introduction
2. Constructible isocrystals
3. The trace map
4. Dual constructibility

$\mathfrak{P}$ = smooth, proper formal $\mathcal{V}$-scheme, $P$ = special fibre,

$$p_1, p_2 : ]P[_{\mathfrak{P}^2} \to \mathfrak{P}_K$$

the two projections.

**Definition (Berthelot)**

A convergent stratification on an $\mathcal{O}_{\mathfrak{P}_K}$-module $E$ is an isomorphism

$$p_2^* E \xrightarrow{\sim} p_1^* E$$

satisfying the cocycle condition.

**Definition (Le Stum)**

$E$ is called constructible if there exists a stratification $P = \coprod_i P_i$ such that $E|_{]P_i[}$ is coherent.

$\mathrm{Isoc}^\dagger_{\mathrm{cons}}(P/K)$ = ($F$-able) constructible $\mathcal{O}_{\mathfrak{P}_K}$-modules with convergent stratification.

**Example**

$X \hookrightarrow \mathfrak{P}$ locally closed immersion, with closure $\alpha : Y \hookrightarrow \mathfrak{P}$, then we have $]\alpha[ : ]Y[ \to \mathfrak{P}_K$, and if $E \in \mathrm{Isoc}^\dagger(X/K) \subset \mathrm{MIC}(j_X^\dagger \mathcal{O}_{]Y[})$, then

$$]\alpha[_! E \in \mathrm{Isoc}^\dagger_{\mathrm{cons}}(P/K)$$

so we have a fully faithful functor

$$]\alpha[_! : \mathrm{Isoc}^\dagger(X/K) \to \mathrm{Isoc}^\dagger_{\mathrm{cons}}(P/K).$$

**Conjecture (Le Stum)**

$\mathbf{R}\mathrm{sp}_*$ *induces an equivalence of categories*

$$\mathrm{Isoc}^\dagger_{\mathrm{cons}}(P/K) \overset{\sim}{\to} \mathrm{Perv}(\mathscr{D}^\dagger_{\mathfrak{P}\mathbb{Q}}).$$

This is a theorem when $\dim \mathfrak{P}/\mathcal{V} = 1$.

Not clear how to even define $\mathbf{R}\mathrm{sp}_*$ in general: want a lifting

$$\begin{array}{ccc} & & D^b(\mathscr{D}^\dagger_{\mathfrak{P}\mathbb{Q}}) \\ & \nearrow & \downarrow \text{forget} \\ \mathrm{Isoc}^\dagger_{\mathrm{cons}}(P/K) & \xrightarrow[\mathbf{R}\mathrm{sp}_*]{} & D^b(\mathscr{D}_{\mathfrak{P}\mathbb{Q}}) \end{array}$$

but $\mathrm{sp}^{-1}\mathscr{D}^\dagger_{\mathfrak{P}\mathbb{Q}}$ doesn't act on constructible isocrysals (even if they are coherent on all of $\mathfrak{P}_K$). What we did: given $X \hookrightarrow Y \overset{\alpha}{\hookrightarrow} \mathfrak{P}$, construct

$$\begin{array}{ccccc} & & & & D^b(\mathscr{D}^\dagger_{\mathfrak{P}\mathbb{Q}}) \\ & & & \nearrow & \downarrow \text{forget} \\ \mathrm{Isoc}^\dagger(X/K) & \xrightarrow[]\alpha[_!]{} & \mathrm{Isoc}^\dagger_{\mathrm{cons}}(P/K) & \xrightarrow[\mathbf{R}\mathrm{sp}_*]{} & D^b(\mathscr{D}_{\mathfrak{P}\mathbb{Q}}) \end{array}$$

This immediately gives

$$H^i_{\mathrm{rig}}(P, ]\alpha[_! E) \cong H^i_{\mathscr{D}}(\mathfrak{P}, \mathbf{R}\mathrm{sp}_* ]\alpha[_! E).$$

Basic example: $\mathfrak{P} = \widehat{\mathbb{P}}^2_{\mathcal{V}}$ with co-ordinates $x_0, x_1, x_2$, $Y = \mathbb{P}^1_k = V(x_2) \subset P$, and $X = \mathbb{A}^1_k = D(x_0) \subset Y$, so we have

$$X \overset{j}{\hookrightarrow} Y \overset{\alpha}{\hookrightarrow} \mathfrak{P}$$

and

$$]\alpha[ : ]Y[ \to \mathfrak{P}_K.$$

We take $E = j_X^\dagger \mathcal{O}_{]Y[_{\mathfrak{P}}}$.

Set $U = P \setminus Y$, so for any sheaf $\mathscr{F}$ on $\mathfrak{P}_K$ we have the localisation exact sequence

$$0 \to \underline{\Gamma}_Y^\dagger \mathscr{F} \to \mathscr{F} \to j_U^\dagger \mathscr{F} \to 0,$$

note that $\underline{\Gamma}_Y^\dagger = ]\alpha[_! ]\alpha[^{-1}$. We apply this to $\mathscr{F} = \mathbf{R}]\alpha[_* j_X^\dagger \mathcal{O}_{]Y[} = ]\alpha[_* j_X^\dagger \mathcal{O}_{]Y[}$ to obtain

$$0 \to ]\alpha[_! j_X^\dagger \mathcal{O}_{]Y[} \to ]\alpha[_* j_X^\dagger \mathcal{O}_{]Y[} \to j_U^\dagger ]\alpha[_* j_X^\dagger \mathcal{O}_{]Y[} \to 0$$

which gives a 2-term resolution of $]\alpha[_! j_X^\dagger \mathcal{O}_{]Y[}$.

**Lemma**

*The sheaves* $]\alpha[_* j_X^\dagger \mathcal{O}_{]Y[}$ *and* $j_U^\dagger ]\alpha[_* j_X^\dagger \mathcal{O}_{]Y[}$ *are* $\mathrm{sp}_*$*-acyclic.*

So we have

$$\mathbf{R}\mathrm{sp}_* ]\alpha[_! j_X^\dagger \mathcal{O}_{]Y[} \cong \left( \mathrm{sp}_* ]\alpha[_* j_X^\dagger \mathcal{O}_{]Y[} \to \mathrm{sp}_* j_U^\dagger ]\alpha[_* j_X^\dagger \mathcal{O}_{]Y[} \right).$$

If we set $u = x_1/x_0$ and $v = x_2/x_0$, and look at global sections, then the first term consists of series

$$f(u,v) \in K[\![u,v]\!]$$

such that:

- $\forall \eta < 1 \ \exists \lambda > 1$ s.t. $f(u,v)$ converges for $|v| \leq \eta$ and $|u| \leq \lambda$.

Can describe the second term similarly, as series

$$f(u,v) \in K[\![u,v,v^{-1}]\!]$$

such that:

- there exists $\rho < 1$ such that $\forall \rho < \eta < 1 \ \exists \lambda > 1$ s.t. $f(u,v)$ converges for $\rho \leq |v| \leq \eta$ and $|u| \leq \lambda$.

Explicitly, the second is Kedlaya's relative Robba ring $\mathcal{R}^{v}_{K\langle u\rangle^\dagger}$, and the first is it's plus part $\mathcal{R}^{v,+}_{K\langle u\rangle^\dagger}$ consisting of series with terms in non-negative powers of $v$.

$\Rightarrow$ can see directly that

$$\mathbf{R}\mathrm{sp}_*]\alpha[_! j_X^\dagger \mathcal{O}_{]Y[} \cong v^{-1}K\langle u, v^{-1}\rangle^\dagger[-1]$$

and the $\mathscr{D}_{\mathfrak{P}\mathbb{Q}}$-module structure extends to a $\mathscr{D}^\dagger_{\mathfrak{P}\mathbb{Q}}$-module structure.

Want to generalise this to an arbitrary frame $(X \overset{j}{\hookrightarrow} Y \overset{\alpha}{\hookrightarrow} \mathfrak{P})$ with $\mathfrak{P}$ smooth and proper over $\mathcal{V}$, and $E \in \mathrm{Isoc}^\dagger(X/K)$.

Three complications:

1. The complement $Y \setminus X$ might not be a divisor. So we take a suitable open cover $X = \cup_a X_a$ and replace $E$ by

$$0 \to \oplus_a j^\dagger_{X_a} E \to \oplus_{a_1,a_2} j^\dagger_{X_{a_1} \cap X_{a_2}} E \to \ldots \to j^\dagger_{\cap_a X_a} E \to 0.$$

2. We don't know in general that the $j^\dagger_{X_a} E$ are $]\alpha[_*$-acyclic. So we take the immersions $\alpha_\eta : [Y]_\eta \to ]Y[$ of quasi-compact tubes, and replace $j^\dagger_{X_a} E$ by

$$\varinjlim_{n_0} \left( \prod_{n \geq n_0} \alpha_{\eta_n *} j^\dagger_{X_a} E|_{[Y]_{\eta_n}} \stackrel{\text{res}-\text{id}}{\longrightarrow} \prod_{n \geq n_0} \alpha_{\eta_n *} j^\dagger_{X_a} E|_{[Y]_{\eta_n}} \right)$$

for $\eta_n \to 1^-$.

3. $Y$ might not be a divisor in $P$, so we need to pick divisors $D_b$ such that $Y = \cap_b D_b$ and replace the short exact sequence

$$0 \to \underline{\Gamma}^\dagger_Y \mathscr{F} \to \mathscr{F} \to j^\dagger_U \mathscr{F} \to 0$$

by the long exact sequence

$$0 \to \underline{\Gamma}^\dagger_Y \mathscr{F} \to \mathscr{F} \to \oplus_b j^\dagger_{U \setminus D_b} \mathscr{F} \to \ldots \to j^\dagger_{U \setminus \cup_b D_b} \mathscr{F} \to 0.$$

**Proposition**

*Given suitable choices $X = \cup_a X_a$ and $Y = \cap_b D_b$ as above there exists a resolution $\mathcal{RC}^\dagger(E)$ of $]\alpha[_! E$ such that the $\mathscr{D}_{\mathfrak{P}\mathbb{Q}}$-module structure on $\mathrm{sp}_* \mathcal{RC}^\dagger(E)$ extends canonically to a $\mathscr{D}^\dagger_{\mathfrak{P}\mathbb{Q}}$-module structure.*

Changing the $X_a$ or the $D_b$ results in canonically quasi-isomorphic complexes of $\mathscr{D}^\dagger_{\mathfrak{P}\mathbb{Q}}$-modules.

**Corollary**

*There exists a canonical lifting of $(\mathbf{R}\mathrm{sp}_* \circ ]\alpha[_!)[d_{\mathfrak{P}}]$ to a functor*

$$\mathbf{R}\mathrm{sp}_{\mathfrak{P},!} : \mathrm{Isoc}^\dagger(X/K) \to D^b(\mathscr{D}^\dagger_{\mathfrak{P}\mathbb{Q}})$$

*such that*

$$H^i_{\mathrm{rig}}(P, ]\alpha[_! E) = H^{i-2d_{\mathfrak{P}}}(u_+ \mathbf{R}\mathrm{sp}_{\mathfrak{P},!} E).$$

**Example**

If $Y = P$ and $Y \setminus X$ is a divisor, then $\mathbf{R}\mathrm{sp}_{\mathfrak{P},!} E = \mathrm{sp}_* E[d_{\mathfrak{P}}] = \mathrm{sp}_+ E[d_{\mathfrak{P}}]$.

Suppose we want to interpret

$$H^i_{\mathrm{rig}}(P, ]\alpha[_! E) = H^i(\mathfrak{P}_K, ]\alpha[_!(E \otimes \Omega^\bullet_{]Y[})).$$

Morally, the RHS should be

$$\text{``}H^i_c(]Y[, E \otimes \Omega^\bullet_{]Y[})\text{''}$$

for some suitable definition of $H^i_c$.

So we want to:

1. make sense of $H^i_c$ for rigid analytic varieties (following Huber);
2. use a suitable formalism of the trace map to understand $H^i_c(]Y[, E \otimes \Omega^\bullet_{]Y[})$ via *duality* (following the approaches of Chiarellotto/van der Put to Serre duality for rigid analytic varieties).

**Proposition**

*For every paracompact (taut) morphism $f : V \to W$ of rigid analytic varieties there exists a unique functor*

$$\mathbf{R}f_! : D^b(\mathbf{Ab}(V)) \to D^b(\mathbf{Ab}(W))$$

*such that:*

- *when $f$ is partially proper, $\mathbf{R}f_!$ is the total derived functor of the functor $f_!$ of sections whose support is quasi-compact over $W$ (in particular, when $f$ is proper, $\mathbf{R}f_! = \mathbf{R}f_*$);*
- *when $f$ is an open immersion, $\mathbf{R}f_! = f_!$ is extension by zero;*
- $\mathbf{R}(g \circ f)_! = \mathbf{R}g_! \circ \mathbf{R}f_!$.

*Moreover, for any $w \in W$ there is a canonical isomorphism*

$$(\mathbf{R}f_!-)_w \xrightarrow{\sim} \mathbf{R}\Gamma_c(f^{-1}(w), -).$$

**Remark**

When $f$ is partially proper, we recover the definition given by van der Put.

By considering more general adic spaces, we can *define* $\mathbf{R}f_!$ by these properties: by a theorem of Huber every such $f : V \to W$ has a canonical compactification

$$\begin{array}{ccc} V & \xrightarrow{j} & \overline{V} \\ & \searrow^{f} & \downarrow{\scriptstyle \bar{f}} \\ & & W \end{array}$$

where $j$ is an open immersion and $\bar{f}$ is partially proper. Then define $\mathbf{R}f_! := \mathbf{R}\bar{f}_! \circ j_!$ where $\mathbf{R}\bar{f}_!$ is the total derived functor of sections with quasi-compact support.

Key new input:

**Theorem (Proper base change theorem)**

*Let $f : V \to W$ be a proper morphism of finite dimensional adic spaces, and $\mathscr{F} \in \mathbf{Ab}(V)$. Then for every $w \in W$ the base change map*

$$(\mathbf{R}f_*\mathscr{F})_w \to \mathbf{R}\Gamma(f^{-1}(w), \mathscr{F})$$

*is an isomorphism.*

**Proof.**

Reduce to $W = \mathrm{Spa}\left(L, L^+\right)$ for some affinoid field $(L, L^+)$, $V = \mathbb{P}^1_{(L,L^+)}$, $w =$ closed point of $W$, and $\mathscr{F} =$ constant sheaf $\underline{A}_T$ supported on some closed subset $T \subset \mathbb{P}^1_{(L,L^+)}$. Then use an explicit topological description of $\mathbb{P}^1_{(L,L^+)}$. □

**Theorem**

*There exists a unique way to associate a trace morphism*

$$\mathrm{Tr}_f : \mathbf{R}f_! \Omega^\bullet_{V/W}[2d] \to \mathcal{O}_W$$

*to every smooth, paracompact morphism $f : V \to W$ of rigid analytic varieties, of relative dimension $d$, such that:*

1. $\mathrm{Tr}_f$ *is compatible with composition;*
2. *when $f$ is étale, then $\mathrm{Tr}_f$ is the canonical map*
$$f_!\mathcal{O}_V \to \mathcal{O}_W;$$
3. *when $W = \mathrm{Sp}(R)$ is affinoid, and $f : \mathbb{D}_W(0;1^-) \to W$ is the canonical projection, then $\mathrm{Tr}_f$ is induced by the map*
$$H^1_c(\mathbb{D}_W(0;1^-), \Omega^1_{\mathbb{D}_W(0;1^-)/W}) \cong R\langle z^{-1}\rangle^\dagger d\log z \to R$$
$$\sum_{i\geq 0} r_i z^{-i} d\log z \mapsto r_0$$
*where $z$ is any co-ordinate on $\mathbb{D}_W(0;1^-)$.*

*If $f$ is either a $\mathbb{D}^n(0;1^-)$ or $\mathbb{A}^{n,\mathrm{an}}$-bundle, then $\mathrm{Tr}_f$ is an isomorphism.*

To construct $\mathrm{Tr}_f$:

- Properties (1) and (3) give a trace map for $f : \mathbb{D}^n_W(0;1^-) \to W$, at least given a choice of co-ordinates $z_1, \ldots, z_d$.
- Properties (1) and (2) then give a trace map for $f : \mathbb{D}^n_W(0;1) \to W$, at least given a choice of $z_1, \ldots, z_d$ and a choice of uniformiser $\pi \in \mathcal{V}$ inducing

$$\mathbb{D}^n_W(0;1) \xrightarrow{j_\pi} \mathbb{D}^n_W(0;1^-).$$

- Properties (1) and (2) then give a trace map whenever $W$ and $V$ are affinoids, at least given a choice of factorisation

$$V \xrightarrow{g} \mathbb{D}^n_W(0;1) \to W$$

  with $g$ étale.
- Can construct $\mathrm{Tr}_f$ in general by using descent.

The hard work is in proving independence of all of these choices!

## Corollary

*Let*

$$\begin{array}{ccccc} & & Y' & \longrightarrow & \mathfrak{P}' \\ & \nearrow & \downarrow g & & \downarrow u \\ X & \longrightarrow & Y & \longrightarrow & \mathfrak{P} \end{array}$$

*be a diagram of frames, with $g$ proper and $u$ smooth in a neighbourhood of $X$, and $E \in \mathrm{Isoc}^{\dagger}(X/K)$. Then $\mathrm{Tr}_{]g[}$ induces an isomorphism*

$$\mathbf{R}]g[_! E_{]Y'[_{\mathfrak{P}'}} \otimes \Omega^{\bullet}_{]Y'[_{\mathfrak{P}'}/]Y[_{\mathfrak{P}}}[2d_u] \stackrel{\sim}{\to} E_{]Y[_{\mathfrak{P}}}$$

*where $d_u$ is the relative dimension.*

## Corollary

*Let $(X, Y \stackrel{\alpha}{\to} \mathfrak{P})$ be a smooth and proper frame over $\mathcal{V}$, and $E \in \mathrm{Isoc}^{\dagger}(X/K)$. Then*

$$H^{2d_{\mathfrak{P}}-i}(\mathfrak{P}_K, ]\alpha[_! E \otimes \Omega^{\bullet}_{\mathfrak{P}_K})$$

*only depends on $X$ and not on $Y$ or $\mathfrak{P}$.*

Now, if we take a smooth and proper frame $(X, Y, \mathfrak{P})$ and $E \in \mathrm{Isoc}^\dagger(X/K)$, then we have Berthelot's "Poincaré" pairing

$$E \times \mathbf{R}\underline{\Gamma}_{]X[_\mathfrak{P}} E^\vee \to \mathbf{R}\underline{\Gamma}_{]X[_\mathfrak{P}} \mathcal{O}_{]Y[_\mathfrak{P}}$$

which via the trace map induces a pairing

$$H_c^{2d_\mathfrak{P}-i}(]Y[_\mathfrak{P}, E \otimes \Omega^\bullet_{]Y[_\mathfrak{P}}) \times H^i(]Y[_\mathfrak{P}, \mathbf{R}\underline{\Gamma}_{]X[_\mathfrak{P}}(E^\vee \otimes \Omega^\bullet_{]Y[_\mathfrak{P}})) \\ \to H_c^{2d_\mathfrak{P}}(]Y[_\mathfrak{P}, \Omega^\bullet_{]Y[_\mathfrak{P}}) \stackrel{\mathrm{Tr}}{\to} K.$$

**Theorem**

*This pairing is perfect.*

**Proof.**

Both sides sit in excision exact sequences, which are compatible with the pairing, so we may assume that $X$ is smooth and affine. We can therefore choose a Monsky–Washnitzer frame $(X, Y, \mathfrak{P})$ in which case the claim reduces to Poincaré duality with coefficients, as proved by Kedlaya. □

So $H^{2d_P - i}_{\mathrm{rig}}(P, ]\alpha[_! E)$ is *canonically* isomorphic to rigid Borel–Moore homology $H^{\mathrm{BM}}_{i,\mathrm{rig}}(X, E)$.

$V$ = complex variety $\rightsquigarrow$ 2 well-known $t$-structures of $D^b_c(V,\mathbb{C})$.

1. The usual (constructible) $t$-structure $(D^{\geq 0}, D^{\leq 0})$ with heart $\mathrm{Con}(V,\mathbb{C})$.
2. The perverse $t$-structure $({}^pD^{\geq 0}, {}^pD^{\leq 0})$ with heart $\mathrm{Perv}(V,\mathbb{C})$.

Second is self-dual under $\mathbf{D}_V$, first is not.

**Definition**

The dual constructible $t$-structure $({}^dD^{\geq 0}, {}^dD^{\leq 0})$ on $D^b_c(V,\mathbb{C})$ is defined by

$$\mathscr{K}^\bullet \in {}^dD^{\geq 0} \Leftrightarrow \mathbf{D}_V(\mathscr{K}^\bullet) \in D^{\leq 0}$$

$$\mathscr{K}^\bullet \in {}^dD^{\leq 0} \Leftrightarrow \mathbf{D}_V(\mathscr{K}^\bullet) \in D^{\geq 0}.$$

Deduce properties of $({}^dD^{\geq 0}, {}^dD^{\leq 0})$ from those of $(D^{\geq 0}, D^{\leq 0})$.

**Example**

If $f : V \to W$ then $f^!$ is exact for the dual constructible $t$-structure. If $f$ is an immersion, then so is $\mathbf{R}f_*$.

Back to char $p$: $\rightsquigarrow D^b_{\mathrm{hol}}(\mathscr{D}^\dagger_{\mathfrak{P}\mathbb{Q}})$, $D^b_{\mathrm{hol}}(X/K)$ also have 3 $t$-structures:

1. holonomic $t$-structure - on $D^b_{\mathrm{hol}}(\mathscr{D}^\dagger_{\mathfrak{P}\mathbb{Q}})$ this is just the obvious one coming from $D^b_{\mathrm{coh}}(\mathscr{D}^\dagger_{\mathfrak{P}\mathbb{Q}})$, slightly more subtle on $D^b_{\mathrm{hol}}(X/K)$;
2. constructible (perverse) $t$-structure;
3. dual constructible $t$-structure.

Same exactness properties as before, in particular the dual constructible $t$-structure on $D^b_{\mathrm{hol}}(X/K)$ is the restriction of that on $D^b_{\mathrm{hol}}(\mathscr{D}^\dagger_{\mathfrak{P}\mathbb{Q}})$ - this is false for the other two!

**Remark**

When $\mathfrak{P}$ is a smooth and proper curve, Le Stum's perverse $t$-structure on $D^b_{\mathrm{hol}}(\mathscr{D}^\dagger_{\mathfrak{P}\mathbb{Q}})$ coincides with our dual constructible $t$-structure, up to a shift by $1 = \dim \mathfrak{P}$.

Hearts are denoted

$$\mathrm{Hol}(\mathfrak{P}),\ \ \mathrm{Con}(\mathfrak{P}),\ \ \mathrm{DCon}(\mathfrak{P})$$

and

$$\mathrm{Hol}(X/K),\ \ \mathrm{Con}(X/K),\ \ \mathrm{DCon}(X/K)$$

respectively.

## Theorem

*Let $X \hookrightarrow \mathfrak{P}$ with $\mathfrak{P}$ smooth and proper over $\mathcal{V}$, and $E \in \mathrm{Isoc}^\dagger(X/K)$. Then*

$$\mathbf{R}\mathrm{sp}_{\mathfrak{P},!}E \in \mathrm{DCon}(X/K) \subset D^b(\mathscr{D}^\dagger_{\mathfrak{P}\mathbb{Q}})$$

*is overholonomic, supported on $X$, and is in the heart of the dual constructible t-structure.*

## Proof.

1. Show that formation of $\mathbf{R}\mathrm{sp}_{\mathfrak{P},!}E$ is compatible with localisation exact sequences and taking finite étale covers of $X$ (this uses a suitable $\mathscr{D}^\dagger$-lifting of the trace morphism).
2. Use alterations to reduce to the case where $X$ and $Y := \overline{X}$ are smooth and $Y \setminus X$ is a divisor.
3. Now locally lift $Y \hookrightarrow \mathfrak{P}$ to a closed embedding $u : \mathfrak{Y} \hookrightarrow \mathfrak{P}$ of smooth formal $\mathcal{V}$-schemes, and show that $u_+\mathbf{R}\mathrm{sp}_{\mathfrak{Y},!}E \cong \mathbf{R}\mathrm{sp}_{\mathfrak{P},!}E$, thus reducing to the case when $Y = \mathfrak{P}_k$.
4. In this case we have $\mathbf{R}\mathrm{sp}_{\mathfrak{P},!} = \mathrm{sp}_+[d_{\mathfrak{P}}]$ and can appeal to Caro–Tsuzuki.

□

## Proposition

*Consider*

$$\begin{array}{ccc} X & \longrightarrow & \mathfrak{P} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle u} \\ Y & \longrightarrow & \mathfrak{Q} \end{array}$$

*with $\mathfrak{P}$, $\mathfrak{Q}$ proper smooth over $\mathcal{V}$, and $E \in \mathrm{Isoc}^\dagger(Y/K)$. Then there is a canonical isomorphism*

$$\mathbf{R}\underline{\Gamma}_X u^! \mathbf{R}\mathrm{sp}_{\mathfrak{Q},!} E \xrightarrow{\sim} \mathbf{R}\mathrm{sp}_{\mathfrak{P},!} f^* E$$

*in $D^b_{\mathrm{hol}}(\mathscr{D}^\dagger_{\mathfrak{P}\mathbb{Q}})$.*

## Proof.

We can treat separately the cases when $u = \mathrm{id}$ and the square is Cartesian. The first follows from compatibility with localisation already mentioned, and the second from direct calculation. □

**Corollary**

*For any variety $X/k$ there exists a canonical functor*

$$\mathrm{sp}_{X,!} : \mathrm{Isoc}^\dagger(X/K) \to \mathrm{DCon}(X/K)$$

*such that for any embedding $X \hookrightarrow \mathfrak{P}$,*

$$(\mathrm{sp}_{X,!}E)_{\mathfrak{P}} = \mathbf{R}\mathrm{sp}_{\mathfrak{P},!}E \in D^b_{\mathrm{hol}}(\mathscr{D}^\dagger_{\mathfrak{P}\mathbb{Q}}).$$

*It is compatible with pullback: for any $f : X \to Y$, and any $E \in \mathrm{Isoc}^\dagger(Y/K)$, we have*

$$\mathrm{sp}_{X,!}f^*E \cong f^!\mathrm{sp}_{Y,!}E.$$

## Theorem

*For any $X/k$, and any $E \in \mathrm{Isoc}^\dagger(X/K)$ there exists a canonical isomorphism*

$$\mathrm{sp}_{X,+}E[-d_X] \xrightarrow{\sim} \mathbf{D}_X(\mathrm{sp}_{X,!}E^\vee)$$

*in $D^b_{\mathrm{hol}}(\mathscr{D}^\dagger_{\mathfrak{P}\mathbb{Q}})$.*

## Proof.

We can show that both sides lie in the abelian category $\mathrm{Con}(X/K)$, which satisfies h-descent. Hence, we may assume that $X$ is smooth, with a smooth compactification $Y$, and that $Y \setminus X$ is a divisor. Then the isomorphism follows from compatibility of Caro's functor $\mathrm{sp}_{X,+}$ with duality. □

## Remark

We only have $\mathrm{sp}_{X,+}E \in \mathrm{Hol}(X/K)$ if $X$ is smooth, in general we have $\mathrm{sp}_{X,+}E \in \mathrm{Con}(X/K)[d_X]$. The formulation of the theorem is slightly neater if we replace $\mathrm{sp}_{X,+}E$ by

$$\widetilde{\mathrm{sp}}_{X,+}E := \mathrm{sp}_{X,+}E[-d_X] \in \mathrm{Con}(X).$$

## Corollary

*For any variety $X/k$, and any $E \in \mathrm{Isoc}^\dagger(X/K)$ we have a canonical isomorphism*

$$H^i_{c,\mathrm{rig}}(X/K,E) \cong H^i_{c,\mathscr{D}}(X/K,E) := H^i(f_! \widetilde{\mathrm{sp}}_{X,+}E)$$

*of $K$-vector spaces.*

## Proof.

If $X \hookrightarrow \mathfrak{P}$ with $\mathfrak{P}$ smooth and proper over $\mathcal{V}$, and $\alpha : Y \hookrightarrow \mathfrak{P}$ is its closure, then we have

$$\begin{aligned} H^i_{c,\mathrm{rig}}(X/K,E) &\overset{\sim}{\to} H^{2d_{\mathfrak{P}}-i}(\mathfrak{P}_K, ]\alpha[_! E^\vee \otimes \Omega^\bullet_{\mathfrak{P}_K})^\vee \\ &\overset{\sim}{\to} H^{d_{\mathfrak{P}}-i}(\mathfrak{P}, \mathbf{R}\mathrm{sp}_{\mathfrak{P},!}E^\vee \otimes \Omega^\bullet_{\mathfrak{P}})^\vee \\ &\overset{\sim}{\to} H^{-i}(f_+ \widetilde{\mathrm{sp}}_{X,!}E^\vee)^\vee \\ &\overset{\sim}{\to} H^i(f_! \widetilde{\mathrm{sp}}_{X,+}E). \end{aligned}$$

The general case can be handled by descent.

□

For comparison of 'usual' cohomologies, see Tomoyuki's talk.

Thank-you!